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Matrix Jacobi Elliptic Hamiltonian and Quasi-Exact Solvability

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Abstract

We construct a new example of 2×2 -matrix quasi-exactly solvable (QES) Hamiltonian which is associated to a potential depending on the Jacobi elliptic functions. We apply the QES analytic method in order to establish three necessary and sufficient algebraic conditions for the previous Hamiltonian to have an invariant vector space whose generic elements are polynomials. This Hamiltonian is called *quasi-exactly solvable*.

Keywords: *Jacobi Elliptic Hamiltonian, QES Analytic Method, Quasi-Exact Solvability*

Résumé

Nous construisons un nouveau type d'hamiltonien partiellement algébrique matriciel d'ordre deux associé à un potentiel dépendant des fonctions elliptiques de Jacobi. Pour ce type d'hamiltonien, seule une partie de valeurs propres peut être déterminée algébriquement. Nous appliquons la méthode analytique de résolubilité partielle pour déterminer les trois conditions algébriques nécessaires et suffisantes pour que le hamiltonien matriciel de Jacobi laisse invariant un espace vectoriel polynômial de dimension finie.

Mots clés: *Hamiltonien Elliptique de Jacobi, Méthode Analytique de Résolubilité Partielle, Résolubilité Partielle*

1. Introduction

In quantum physics, the goal consists in constructing the spectrum of a linear operator defined on a suitable domain of Hilbert space. In most cases, this type of problem cannot be explicitly solved, in other words the eigenvalues of the Hamiltonian cannot be computed algebraically. However, in few cases, some of which turn out to be physically fundamental, the spectrum can indeed be found explicitly. The two major examples of this kind are the celebrated harmonic quantum oscillator and the hydrogen atom (i.e. 3-dimensional Schrödinger equation coupled to an external Coulomb potential). These examples are called *exactly solvable* in the sense that the full spectrum of the Hamiltonian is found explicitly.

In the last few years, a new class of operators which is intermediate to exactly solvable and non solvable operators has been discovered (Turbiner, 1988; Ushveridze, 1995; Cayley, 1961; Turbiner, 1989; Shifman & Turbiner, 1989): *the quasi-exactly solvable* (*QES*) operators, for which a finite part of the spectrum can computed algebraically.

Although scalar QES operators have been classified in one variable (González-López, *et al., 1993)* and in several variables [A. González-López, *et al.*,1991), a classification of matrix QES operators is still missing.

More recently, interesting tools for classification of 2×2-matrix QES operators in one spatial dimensional (Zhdanov, 1997**;** Y. Brihaye & P. Kosinski, 1997; Y. Brihaye et al., 2007) and in creation and annihilation operators (A. Nininahazwe, 2013) have been constructed.

In another work (Y. Brihaye et al., 2007), PT-symmetric, QES 2×2-matrix Hamiltonians are analyzed with the emphasis set on the reality properties of the eigenvalues. The authors considered both trigonometric and hyperbolic 2×2-matrix Hamiltonians. A set of necessary and sufficient conditions (i.e. QES conditions) for 2×2-matrix operators to preserve a vector space of polynomials have been proposed. These QES conditions constitute the so-called QES analytic method. We construct this new example of Hamiltonian in order to put out another method used to prove the quasi-exact solvability property.

This paper is organized as follows: In the section. 2 based on the Refs. (Y. Brihaye et al., 2007; A. Nininahazwe, 2013), we briefly recall the QES analytic method used to investigate the quasi-exact solvability of 2×2-matrix operators. In section.3, along the same lines as in the Refs. (Y. Brihaye et al., 2007; A. Nininahazwe, 2013), we apply the QES analytic method in order to construct a new 2×2-matrix QES Hamiltonian depending on Jacobi elliptic potential. We will consider two values of the constant δ : the case $\delta=1$ and the case $\delta=2$, the interest results will be Computed.

2. QES Analytic Method

Taking account to the same lines as in the Refs. (Y. Brihaye et al., 2007; A. Nininahazwe, 2013; A. Nininahazwe, 2020), we recall a general method to check whether a 2×2-matrix differential operator H (in a variable x) preserves a vector space whose components are polynomials.

Consider the $2\times 2-$matrix Hamiltonian of the following form (Y. Brihaye et al., 2007; A. Nininahazwe, 2013; A. Nininahazwe, 2020):

$$H=\begin{pmatrix} -\frac{d^2}{dx^2}+V_{11}(x) & x^{\delta} \\ x^{\delta'} & -\frac{d^2}{dx^2}+V_{22}(x) \end{pmatrix} \quad (1)$$

where:

$$V(x)=\begin{pmatrix} V_{11}(x) & x^{\delta} \\ x^{\delta'} & V_{22}(x) \end{pmatrix}, \quad \delta=0,1,2,$$

$$\delta'=1-\delta,\ V_{12}(x)=x^{\delta},\ V_{21}(x)=x^{\delta'}$$

$V(x)$ is the potential associated to the Hamiltonian H given by this above relation (1).

A gauge transformation and a change of variable on the Hamiltonian H lead to the following Hamiltonian called the gauge one

$$\tilde{H}=\phi^{-1}H\phi \quad (2)$$

which can be written in his components as follows:

$$\widetilde{H} = \widetilde{H}_1 + \widetilde{H}_0 + \widetilde{H}_{-1} \quad (3)$$

More precisely, the diagonal components of $\widetilde{H}_1$ are differential operators and the off-diagonal components $(\widetilde{H}_1)_{12}$ and $(\widetilde{H}_1)_{21}$ are respectively proportional to x^{δ} and $x^{\delta'}$. The operators $\widetilde{H}_0$ and $\widetilde{H}_{-1}$ have lower degrees in all their components than the corresponding components in $\widetilde{H}_1$. Note that the invariant vector space of the Hamiltonian $\widetilde{H}$ has the following form (Y. Brihaye et al., 2007; A. Nininahazwe, 2013; A. Nininahazwe, 2020; A. Nininahazwe, 2018):

$$W = \left\{ \begin{pmatrix} p_n \\ q_n \end{pmatrix} \right\}, \quad (4)$$

$$m = n - \delta + 1, \, m \in \mathbb{N}$$

In order to obtain the QES conditions for $\widetilde{H}$, the generic vector of the above vector space is of the form

$$\psi = \begin{pmatrix} \alpha_0 x^n + \alpha_1 x^{n-1} \\ \beta_0 x^{n-\delta+1} + \beta_1 x^{n-\delta} \end{pmatrix}, \quad (5)$$

where $\alpha_i, \beta_i \; (i = 0,1)$ are complex parameters. As a consequence the 2×2-matrix $M_1, \widetilde{M}_1, M_0$ are defined by:

$$\widetilde{H}_1 \begin{pmatrix} \alpha_0 x^n \\ \beta_0 x^{n-\delta+1} \end{pmatrix} = diag\left(x^{n+1}, \quad x^{n-\delta+2}\right) M_1 \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix},$$

$$\widetilde{H}_1 \begin{pmatrix} \alpha_1 x^{n-1} \\ \beta_1 x^{n-\delta} \end{pmatrix} = diag\left(x^n, \quad x^{n-\delta+1}\right) \widetilde{M}_1 \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix},$$

$$\widetilde{H}_0 \begin{pmatrix} \alpha_0 x^n \\ \beta_0 x^{n-\delta+1} \end{pmatrix} = diag\left(x^n, \quad x^{n-\delta+1}\right) M_0 \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} \quad (6)$$

The three necessary and sufficient QES conditions for $\widetilde{H}$ to have an invariant vector space are

i) $M_1 \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,

ii) $\widetilde{M}_1{}^t \begin{pmatrix} -\beta_0 \\ \alpha_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,

iii) $M_0 \begin{pmatrix} 1 \\ \frac{\beta_0}{\alpha_0} \end{pmatrix} = \Lambda \begin{pmatrix} 1 \\ \frac{\beta_0}{\alpha_0} \end{pmatrix}$. (7)

In the next step, we will apply in a same lines of this QES analytic method in order to prove the quasi-exact solvability of the 2×2-matrix QES Hamiltonian associated to Jacobi Elliptic Potential.

3. QES Jacobi Hamiltonian

3.1. Case δ = 1

In this section, we apply the QES analytic method established in previous section to check whether the 2×2-matrix operator is quasi-exactly solvable. We consider the 2×2-matrix Hamiltonian depending on the Jacobi elliptic potential of the form (Y. Brihaye et al., 2007; A. Nininahazwe, 2013; A. Nininahazwe, 2020):

$$H(z) = -\frac{d^2}{dz^2} 1_2 + V_D + V_I \quad (8)$$

with

$$V_D = sn^2(z,k) diag(a_1, a_2) + diag(b_1, b_2) \, ,$$

$$\mathbf{V_D} = \begin{pmatrix} sn^2 a_1 + b & 0 \\ 0 & sn^2 a_2 - b \end{pmatrix} \quad (9)$$

and 1_2 is the matrix identity , $a_1, a_2, b_1, b_2, \theta$ denote real constants and V_I is symmetric off-diagonal matrix of the form

$$V_I(z) = \begin{pmatrix} 0 & \theta sndn \\ \theta sndn & 0 \end{pmatrix} \quad (10)$$

Note that the above Hamiltonian is to be considered on the Hilbert space of periodic functions on [0, 4K(k)].

Note that the sum $V_D + V_I$ is the potential associated to the Hamiltonian H(z) .

Using the following change of function (i.e. the gauge transformation), the gauge Hamiltonian is written as follows:

$$\tilde{H}(z) = h^{-1}H(z)h,$$

$$\tilde{H}(z) = \begin{pmatrix} \tilde{H}_{11} & \tilde{H}_{12} \\ \tilde{H}_{21} & \tilde{H}_{22} \end{pmatrix} \qquad (11)$$

where

$$\tilde{H}_{11} = -\frac{d^2}{dz^2} - 2\frac{h_1'}{h_1}\frac{d}{dz} - \frac{h_1''}{h_1} + a_1 sn^2 + b,$$

$$\tilde{H}_{12} = \theta dn^2,$$

$$\tilde{H}_{21} = \theta sn^2,$$

$$\tilde{H}_{22} = -\frac{d^2}{dz^2} - 2\frac{h_2'}{h_2}\frac{d}{dz} - \frac{h_2''}{h_2} + a_2 sn^2 - b \quad (12)$$

and

$$h = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix},$$

$$h = \begin{pmatrix} sn(z,k) & 0 \\ 0 & dn(z,k) \end{pmatrix} \qquad (13)$$

The relevant change of variable consists in posing

$$t = sn^2(z,k).$$

Taking account to the reference (A. Nininahazwe, 2013; A. Nininahazwe, 2020), the differential symbol $\frac{d^2}{dz^2}$ has the following form

$$\frac{d^2}{dz^2} = 4t(1-t)(1-k^2t)\frac{d^2}{dt^2} + 2(3k^2t^2 - 2(1+k^2)t + 1)\frac{d}{dt} \quad (14)$$

We recall that for generic values of k, the Jacobi functions obey the following relations (A. Nininahazwe, 2013; A. Nininahazwe, 2020):

$$cn^2 + sn^2 = 1,$$

$$dn^2 + k^2 sn^2 = 1$$

$$\frac{d}{dz}sn = cndn,$$

$$\frac{d}{dz}sn^2 = 2sncndn$$

$$\frac{d}{dz}cn = -sndn,$$

$$\frac{d}{dz}dn = -k^2 sncn \qquad (15)$$

The following identities are used to establish the gauge Hamiltonian (11) in the variable $t = sn^2(z,k)$ (A. Nininahazwe, 2013; A. Nininahazwe, 2020):

h	$\frac{h''}{h}$	$\frac{h'}{h}(sncndn)$
1	0	0
sn	$2k^2t-(1+k^2)$	$k^2t^2-(1+k^2)t+1$
cn	$2k^2t-1$	k^2t^2-t
dn	$2k^2t-k^2$	$k^2t^2-k^2t$
$cndn$	$6k^2t-(1+k^2)$	$2k^2t^2-(1+k^2)t$
$sndn$	$6k^2t-(1+4k^2)$	$2k^2t^2-(1+2k^2)t+1$
$sncn$	$6k^2t-(4+k^2)$	$2k^2t^2-(2+k^2)t+1$
$sncndn$	$12k^2t-4(1+k^2)$	$3k^2t^2-2(1+k^2)t+1$ (16)

Referring to the above relations (16), for $h_1 = sn$, the second term and the third term of the operator $\tilde{H}_{11}$ of the equation (12) are written as follows:

$$-2\frac{h_1'}{h_1}\frac{d}{dz} = -2\frac{h_1'}{h_1}(sncndn)\frac{d}{dt},$$

$$-2\frac{h_1'}{h_1}\frac{d}{dz} = -4[k^2t^2 - (1+k^2)t + 1]\frac{d}{dt}, \quad (17)$$

$$-\frac{h_1''}{h_1} = -2k^2 t + 1 + k^2. \quad (18)$$

Referring to the same identities given by the equation (16), for $h_2 = dn$, the second term and the third term of the operator $\tilde{H}_{22}$ of the equation (12) are of the following form:

$$-2\frac{h_2'}{h_2}\frac{d}{dz} = -2\frac{h_2'}{h_2}(2sncndn)\frac{d}{dt},$$

$$-2\frac{h_2'}{h_2}\frac{d}{dz} = -4(k^2t^2 - k^2t)\frac{d}{dt}, \quad (19)$$

$$-\frac{h_2''}{h_2} = -2k^2 t + k^2 \quad (20)$$

Considering the change of variable $t = sn^2(z,k)$, the fourth and fifth terms of the components $\tilde{H}_{11}$ and $\tilde{H}_{22}$ of the gauge Hamiltonian $\tilde{H}$ are respectively rewritten as follows:

$$\begin{pmatrix} a_1 sn^2 + b & 0 \\ 0 & a_2 sn^2 - b \end{pmatrix} = \begin{pmatrix} a_1 t + b & 0 \\ 0 & a_2 t - b \end{pmatrix} \quad (21)$$

Taking account of the change of variable $t = sn^2(z,k)$ the identities and $dn^2 + k^2 sn^2 = 1$ lead respectively to new form of the two off-diagonal components of the gauge Hamiltonian given by the equation (12):

$$\tilde{H}_{12} = \theta dn^2,$$

$$\tilde{H}_{12} = \theta - \theta k^2 t \quad (22)$$

$$\tilde{H}_{21} = \theta sn^2,$$

$$\tilde{H}_{21} = \theta t, \quad (23)$$

Replacing the terms of the components of the Hamiltonian $\tilde{H}$ given by the equation (12) by the expressions (14) and (17)-(23), one can easily write (in variable t) the gauge Hamiltonian as follows:

$$\tilde{H}_{11} = -4t(1-t)(1-k^2t)\frac{d^2}{dt^2} - 2[5k^2t^2 - 4(1+k^2)t + 3]\frac{d}{dt} + a_1 t - 2k^2 t + b + 1 + k^2,$$

$$\tilde{H}_{12} = \theta(1-k^2t),$$

$$\tilde{H}_{21} = \theta t,$$

$$\tilde{H}_{22} = -4t(1-t)(1-k^2t)\frac{d^2}{dz^2} - 2[5k^2t^2 - 2(1+2k^2)t + 1]\frac{d}{dt} + a_2 t - 2k^2 t + k^2 - b. \quad (24)$$

The next step is to establish the QES conditions of the gauge Hamiltonian. In other words, we put out the expressions of the real parameters a_1, b and θ. Let us express the gauge Hamiltonian $\tilde{H}$ given by the above relations (24) in its components according to

$$\tilde{H} = \tilde{H}_1 + \tilde{H}_0 + \tilde{H}_{-1} \quad (25)$$

where:

$$\tilde{H}_1=\begin{pmatrix}-4k^2t^3\frac{d^2}{dt^2}-10k^2t^2\frac{d}{dt}+(a_1-2k^2)t & -\theta k^2t\\ \theta t & -4k^2t^3\frac{d^2}{dt^2}-10k^2t^2\frac{d}{dt}+(a_2-2k^2)t\end{pmatrix},$$

$$\tilde{H}_0=\begin{pmatrix}(4k^2+4)t^2\frac{d^2}{dt^2}+8(1+k^2)t\frac{d}{dt}+1+b+k^2 & \theta\\ 0 & (4k^2+4)t^2\frac{d^2}{dt^2}+4(1+2k^2)t\frac{d}{dt}+k^2-b\end{pmatrix},$$

$$\tilde{H}_{-1}=\begin{pmatrix}-4t\frac{d^2}{dt^2}-6\frac{d}{dt} & 0\\ 0 & -4t\frac{d^2}{dt^2}-2\frac{d}{dt}\end{pmatrix} \quad (26)$$

As $\delta=1$, the generic wave function ψ given by the equation (5) is written as follows:

$$\psi=\begin{pmatrix}\alpha_0t^n+\alpha_1t^{n-1}+...\\ \beta_0t^n+\beta_1t^{n-1}+..\end{pmatrix} \quad (27)$$

$$\tilde{H}_1\begin{pmatrix}t^n\\ t^n\end{pmatrix}\cong\begin{pmatrix}t^{n+1}\\ t^{n+1}\end{pmatrix},$$

$$\tilde{H}_0\begin{pmatrix}t^n\\ t^n\end{pmatrix}\cong\begin{pmatrix}t^{n}\\ t^{n}\end{pmatrix},$$

$$\tilde{H}_{-1}\begin{pmatrix}t^n\\ t^n\end{pmatrix}\cong\begin{pmatrix}t^{n-1}\\ t^{n-1}\end{pmatrix}. \quad (28)$$

After some algebraic manipulations, one can easily obtain the $2\times2-$matrices $M_1,\tilde{M}_1,M_0$ respectively as follows (Y. Brihaye et al., 2007; A. Nininahazwe, 2013; A. Nininahazwe, 2020):

Note that the action of these above three gauge components of $\tilde{H}$ given by the relations (26) on the wave function ψ given by the relation (27) leads to the following expressions:

$$\tilde{H}_1\begin{pmatrix}\alpha_0t^n\\ \beta_0t^n\end{pmatrix}=diag\left(t^{n+1},t^{n+1}\right)M_1\begin{pmatrix}\alpha_0\\ \beta_0\end{pmatrix},$$

$$\tilde{H}_1\begin{pmatrix}\alpha_1t^{n-1}\\ \beta_1t^{n-1}\end{pmatrix}=diag\left(t^{n},t^{n}\right)\tilde{M}_1\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix},$$

$$\tilde{H}_0\begin{pmatrix}\alpha_0t^n\\ \beta_0t^n\end{pmatrix}=diag\left(t^{n},t^{n}\right)M_0\begin{pmatrix}\alpha_0\\ \beta_0\end{pmatrix}. \quad (29)$$

Taking account to these above expressions given by the equation (29), one can easily find the following matrices:

$$M_1=\begin{pmatrix}-4k^2n(n-1)-10nk^2+a_1-2k^2 & -\theta k^2\\ \theta & -4k^2n(n-1)-10nk^2+a_2-2k^2\end{pmatrix},$$

$$\tilde{M}_1=\begin{pmatrix}-4k^2n(n-1)(n-2)-10k^2(n-1)+a_1-2k^2 & -\theta k^2\\ \theta & -4k^2n(n-1)(n-2)-10k^2(n-1)+a_2-2k^2\end{pmatrix}$$

$$M_0=\begin{pmatrix}n(n-1)(4k^2+4)+8(1+k^2)n+1+b+k^2 & \theta\\ 0 & n(n-1)(4k^2+4)+4(1+2k^2)n+k^2-b\end{pmatrix} \quad (30)$$

Taking account to the above relations (7), the three necessary and sufficient QES conditions for the operator $\tilde{H}$ to have a finite dimensional invariant vector space vector are obtained:

i) The first QES condition is:

$$M_1\begin{pmatrix}\alpha_0\\ \beta_0\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix},$$
$$\det M_1 = 0,$$

$$\theta^2 = -16k^2n^4 - 48k^2n^3 - 52k^2n^2 - 24k^2n - 4k^4 + \left(+4n^2+6n+2\right)(a_1+a_2) - \frac{a_1a_2}{k^2} \qquad (31)$$

ii) The second QES condition is as follows:

$$\tilde{M}_1^{\,t}\begin{pmatrix}-\beta_0\\ \alpha_0\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix},$$
$$\det\tilde{M}_1 = 0,$$

$$\det\tilde{M}_1 = [-4k^2n(n-1)(n-2)-10k^2(n-1)+a_1-2k^2][-4k^2n(n-1)(n-2)-10k^2(n-1)+a_2-2k^2]-\theta^2k^2$$

In this above equation replacing θ^2 by its value (31) and after some algebraic manipulations, the second QES condition is obtained:

$$a_1 = \frac{32k^2n^3+24k^2n^2+12k^2n+2k^2-4na_2-a_2}{4n+1} \qquad (32)$$

iii) The final and the third QES condition is computed as follows

$$M_0\begin{pmatrix}1\\ \dfrac{\beta_0}{\alpha_0}\end{pmatrix}=\Lambda\begin{pmatrix}1\\ \dfrac{\beta_0}{\alpha_0}\end{pmatrix}, \qquad (33)$$

where Λ is a constant and

$$\frac{\beta_0}{\alpha_0} = \frac{-4k^2n^2-6k^2n+a_1-2k^2}{\theta k^2} \qquad (34)$$

Referring to the expression of the matrix M_0 given by the equation (30) and the relation (34), the relation (33) leads to the following third QES condition:

$$b = \frac{4k^2n^2+2k^2n+k^2-a_1}{2k^2} \qquad (35)$$

Taking account to the above QES conditions given by the equations (31), (32) and (35), we can conclude that the Hamiltonian $\tilde{H}$ (therefore H) is quasi-exactly solvable (Y. Brihaye et al., 2007; A. Nininahazwe, 2013; A. Nininahazwe, 2020). In other words, a finite part of the spectrum of the Hamiltonian $\tilde{H}$ can be computed algebraically.

3.2. Case δ = 2

Along the same lines applied for the previous case (i.e. for the case δ = 1), we perform a gauge transformation according to

$$\tilde{H}(z) = r^{-1}H(z)r,$$
$$\tilde{H}(z) = \begin{pmatrix}\tilde{H}_{11} & \tilde{H}_{12}\\ \tilde{H}_{21} & \tilde{H}_{22}\end{pmatrix} \qquad (36)$$

with

$$r = \begin{pmatrix}r_1 & 0\\ 0 & r_2\end{pmatrix}, \quad r_1 = cn, \quad r_2 = sncndn,$$
$$r = \begin{pmatrix}cn & 0\\ 0 & sncndn\end{pmatrix},$$

$$r^{-1} = \begin{pmatrix}\dfrac{1}{cn} & 0\\ 0 & \dfrac{1}{sncndn}\end{pmatrix},$$

After gauge transformation on the Hamiltonian H given by the equation (8), the components of the above Hamiltonian $\tilde{H}$ of the relation (36) are written as follows

$$\tilde{H}_{11} = -\frac{d^2}{dz^2} - 2\frac{r_1'}{r_1}\frac{d}{dz} - \frac{r_1''}{r_1} + a_1sn^2 + b,$$
$$\tilde{H}_{12} = \theta sn^2dn^2,$$
$$\tilde{H}_{21} = \theta,$$

$$\tilde{H}_{22} = -\frac{d^2}{dz^2} - 2\frac{r_2'}{r_2}\frac{d}{dz} - \frac{r_2''}{r_2} + a_2 sn^2 - b \quad (37)$$

Referring to the table of identities given by the equation (16), the second term and the third term of the component operator $\tilde{H}_{11}$ (37) are of the following form

$$-2\frac{r_1'}{r_1}\frac{d}{dz} = -4\frac{r_1'}{r_1}(sncndn)\frac{d}{dt},$$

$$-2\frac{r_1'}{r_1}\frac{d}{dz} = -4t(k^2t-1)\frac{d}{dt}, \quad (38)$$

$$-\frac{r_1''}{r_1} = -2k^2t + 1 \quad (39)$$

with $r_1 = cn$. For $r_2 = sncndn$, taking account to the same identities (16), the following second and the third terms of the component operator $\tilde{H}_{22}$ given by the relation (37):

$$-2\frac{r_2'}{r_2}\frac{d}{dz} = -4\frac{r_2'}{r_2}(sncndn)\frac{d}{dt}$$

$$-2\frac{r_2'}{r_2}\frac{d}{dz} = -[12k^2t^2 - 8(1+k^2)t + 4]\frac{d}{dt}, \quad (40)$$

$$-\frac{r_2''}{r_2} = -12k^2t + 4(1+k^2). \quad (41)$$

Referring to the change of variable $t = sn^2(z,k)$, the fourth and fifth terms of the components $\tilde{H}_{11}$ and $\tilde{H}_{22}$ of the gauge Hamiltonian $\tilde{H}$ given by the relation (37) have the following form:

$$\begin{pmatrix} a_1 sn^2 + b & 0 \\ 0 & a_2 sn^2 - b \end{pmatrix} = \begin{pmatrix} a_1 t + b & 0 \\ 0 & a_2 t - b \end{pmatrix} \quad (42)$$

Taking account to the change of the variable $t = sn^2(z,k)$ and the relations, $dn^2 + k^2 sn^2 = 1$, the off-diagonal component $\tilde{H}_{12}$ of the gauge Hamiltonian $\tilde{H}$ given by the relation (38) is written as the follows:

$$\tilde{H}_{12} = \theta sn^2 dn^2,$$

$$\tilde{H}_{12} = \theta t(1 - k^2 t) \quad (43)$$

Note that the off-diagonal component $\tilde{H}_{21}$ of the gauge Hamiltonian $\tilde{H}$ given by the relation (38) keeps the same expression

$$\tilde{H}_{21} = \theta \quad (44)$$

Replacing the four components of the gauge Hamiltonian $\tilde{H}(t)$ given by the equation (37) by the expressions (14) and (38)-(44), one can easily found their final form in variable t:

$$\tilde{H}_{11} = -4t\frac{d^2}{dt^2} + 4(k^2+1)t^2\frac{d^2}{dt^2} - 4k^2t^3\frac{d^2}{dt^2} - 10k^2t^2\frac{d}{dt} + (8+4k^2)t\frac{d}{dt} - 2\frac{d}{dt} + (a_1 - 2k^2)t + 1 + k^2 + b$$

$$\tilde{H}_{12} = \theta t - \theta k^2 t^2,$$

$$\tilde{H}_{21} = \theta,$$

$$\tilde{H}_{22} = -4t\frac{d^2}{dt^2} + 4(k^2+1)t^2\frac{d^2}{dt^2} - 4k^2t^3\frac{d^2}{dt^2} - 18k^2t^2\frac{d}{dt} + (12+12k^2)t\frac{d}{dt} - 6\frac{d}{dt} + (a_2 - 12k^2)t + 4 + 4k^2 - b \quad (45)$$

Note that the generic element of the invariant vector space V for $\delta = 2$ is given by the equation (5) as in the QES analytic method:

$$\psi = \begin{pmatrix} \alpha_0 t^n + \alpha_1 t^{n-1} \\ \beta_0 t^{n-\delta+1} + \beta_1 t^{n-\delta} \end{pmatrix},$$

$$\psi = \begin{pmatrix} \alpha_0 t^n + \alpha_1 t^{n-1} \\ \beta_0 t^{n-1} + \beta_1 t^{n-2} \end{pmatrix} \qquad (46)$$

Recall that the action of the gauge components of $\tilde{H}$ on the generic function ψ given by the relation (46) leads to the following expressions:

$$\tilde{H}_1 \begin{pmatrix} t^n \\ t^{n-1} \end{pmatrix} \cong \begin{pmatrix} t^{n+1} \\ t^n \end{pmatrix},$$

$$\tilde{H}_0 \begin{pmatrix} t^n \\ t^{n-1} \end{pmatrix} \cong \begin{pmatrix} t^n \\ t^{n-1} \end{pmatrix},$$

$$\tilde{H}_{-1} \begin{pmatrix} t^n \\ t^{n-1} \end{pmatrix} \cong \begin{pmatrix} t^{n-1} \\ t^{n-2} \end{pmatrix}. \qquad (47)$$

Referring to the above expressions (47), the three components of the gauge Hamiltonian $\tilde{H}$ are deduced

$$\tilde{H}_1 = \begin{pmatrix} -4k^2t^3 \frac{d^2}{dt^2} - 10k^2t^2 \frac{d}{dt} + (a_1 - 2k^2)t & -\theta k^2 t^2 \\ \theta & -4k^2t^3 \frac{d^2}{dt^2} - 18k^2t^2 \frac{d}{dt} + (a_2 - 12k^2)t \end{pmatrix}$$

$$\tilde{H}_0 = \begin{pmatrix} 4(k^2+1)t^2 \frac{d^2}{dt^2} + 4(2+k^2)t \frac{d}{dt} + +1 + b & \theta t \\ 0 & 4(k^2+1)t^2 \frac{d^2}{dt^2} + 12(k^2+1)t \frac{d}{dt} + 4(1+k^2) - b \end{pmatrix}$$

$$\tilde{H}_{-1} = \begin{pmatrix} -4t \frac{d^2}{dt^2} - 2\frac{d}{dt} & \theta \\ 0 & -4t \frac{d^2}{dt^2} - 6\frac{d}{dt} \end{pmatrix} \qquad (48)$$

As it is shown by the relation (47), the above operators $\tilde{H}_1, \tilde{H}_0$ and $\tilde{H}_{-1}$ are respectively the matrix operators which increases, preserves and reduces the degree of the generic element ψ given by the equation (46). As a consequence the vector $\tilde{H}\psi$ can be decomposed as follows:

$$\tilde{H}\psi = diag\left(t^{n+1}, \quad t^n\right) M_1 \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} + diag\left(t^n, \quad t^{n-1}\right) \tilde{M}_1 \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} + diag\left(t^n, \quad t^{n-1}\right) M_0 \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} \qquad (49)$$

where the constant 2×2-matrices $M_1, \tilde{M}_1$ and M_0 can be computed explicitly after a some calculations

$$\tilde{H}_1 \begin{pmatrix} \alpha_0 t^n \\ \beta_0 t^{n-1} \end{pmatrix} = diag\left(t^{n+1}, \quad t^n\right) M_1 \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix}$$

where:

$$M_1 = \begin{pmatrix} -4k^2 n(n-1) - 10k^2 n + a_1 - 2k^2 & -\theta k^2 \\ \theta & -4k^2(n-1)(n-2) - 18k^2(n-1) + a_2 - 12k^2 \end{pmatrix}$$

One can easily deduce the matrix $\tilde{M}_1$ from the following expression

$$\tilde{H}_1\begin{pmatrix}\alpha_1 t^{n-1}\\ \beta_1 t^{n-2}\end{pmatrix} = diag\left(t^n, \quad t^{n-1}\right)\tilde{M}_1\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix}$$

where:

$$\tilde{M}_1 = \begin{pmatrix} -4k^2(n-1)(n-2)-10k^2(n-1)+a_1-2k^2 & -\theta k^2 \\ \theta & -4k^2(n-2)(n-3)-18k^2(n-2)+a_2-12k^2 \end{pmatrix}$$

Finally the third matrix M_0 is easily found as follows: $\tilde{H}_0\begin{pmatrix}\alpha_0 t^n\\ \beta_0 t^{n-1}\end{pmatrix} = diag\left(t^n, \quad t^{n-1}\right)M_0\begin{pmatrix}\alpha_0\\ \beta_0\end{pmatrix}$

where:

$$M_0 = \begin{pmatrix} (4k^2+4)n(n-1)+4n(2+k^2)+1+b+k^2 & \theta \\ 0 & (4k^2+4)(n-1)(n-2)+(12+12k^2)(n-1)+4+4k^2-b \end{pmatrix}$$

Referring to the three QES conditions given by the relations (7) of the QES analytic method and to the expressions of the previous three 2×2-matrices $M_1, \tilde{M}_1$ and M_0, one can easily compute algebraically the three necessary and sufficient conditions for the gauge Hamiltonian $\tilde{H}$ given by the expressions (45) to be quasi-exactly solvable as follows:

i) the first QES condition is as follows

$$\det M_1 = 0,$$

$$\theta^2 = -\left(16n^4+48n^3+52n^2+16n\right)k^2 - 2n\left(3a_1+a_2\right)-2\left(a_1+a_2\right)+4k^2+\frac{a_1 a_2}{k^2} \quad (50)$$

ii) the second QES condition is easily checked

$$a_1 = \frac{32k^2n^3+24k^2n^2+12k^2n+2k^2-(4n+1)a_2}{38k^2+8k^2n} \quad (51)$$

iii) Finally the third QES condition is found

$$b = \frac{4k^2n^2+10k^2n+k^2-a_1}{2k^2} \quad (52)$$

4. Conclusion

In this paper, we have applied the QES analytic method established previously (Y. Brihaye et al., 2007) in order to construct a 2×2-matrix QES Hamiltonian which is associated to a Jacobi elliptic potential. We have considered two cases: $\delta = 1$ and $\delta = 2$, more precisely, the three necessary and sufficient conditions (i.e. the three QES conditions) for the Jacobi elliptic Hamiltonian to be called quasi-exactly solvable are computed algebraically.

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