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2 N Parameters models of finitely many nonrelativistic of δ' sphere and δ' sphere plus coulomb interactions in quantum mechanics

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Abstract

We make a systematic study of new exactly solvable models of $2N$ parameters models of finitely many nonrelativistic δ' -sphere and δ' -sphere plus Coulomb interactions. They constitute a generalization of 2N parameters nonrelativistic δ'-sphere and δ'-sphere plus Coulomb interactions. We provide the mathematical definitions of hamiltonians and obtain new results for both model, in particular the resolvents equations, spectral properties and some scattering quantities theory. When we have an appropriate choice of the parameters linked to the above-mentioned model, we show that the unification of the δ' -sphere interactions of the first and second type leads to an increase in scattering amplitude and a decrease of scattering length.

Keywords: *boundary conditions problem, δ'-sphere interactions, self adjoint operator, resolvent equation, spectral properties, scattering theory*.

Résumé

Nous étudions deux nouveaux modèles à $2N$ paramètres d'interactions δ'-sphériques non-relativistes et δ'-sphériques non relativistes plus une interaction Coulombienne centrées sur un nombre fini de sphères. Ils constituent une généralisation des interactions δ'-sphériques et δ'-sphériques plus une interaction Coulombienne centrées sur une sphère. Nous définissons les hamiltoniens et nous obtenons de nouveaux résultats pour les deux modèles: Equations résolvantes, les propriétés spectrales et les éléments de la théorie de diffusion. Lorsqu' on fait un choix approprié des paramètres liés au modèle cité ci-haut, nous montrons que l'unification des interactions δ' -sphériques de première et de deuxième espèce (qui donne naissance au modèle à $2N$ paramètres) entraine un accroissement de l'amplitude de diffusion et une diminution de la longueur de diffusion.

Keywords: *Conditions au limites, Interaction δ'-spherique, extensions auto adjointes, Equation résolvante, théorie de diffusion*

1. Introduction

The study of sphere interactions is interesting for understanding various physical phenomena in quantum mechanics (Hounkonnou M.N. et al., 1999, Antoine J.P. et al., 1988; Shabani J., 1988; Dabrowski L. & Shabani J., 1988; Antoine J.P et al., 1994; Shabani J., 1988; Hounkonnou M.N. et al., 1997 ; Dittrich J.et al., 1989; Dittrich J. et al., 1992; Shabani J. & Vyabandi A., 2002; Green I.M. & Moszkowski S.A.,1965 ; Blinder S.M., 1978; Rubio J. and Garcia- Moliner F, 1967). To the best of our knowledge, the study of δ'-sphere interactions based on theory of self adjoint extension of symmetric operators in the Hilbert space began in the twentieth century(Hounkonnou M. et al., 1999). And so far little work has been done in this area both in relativistic and non-relativistic quantum mechanics (Hounkonnou M. et al., 1999; Akhiezer W.I. and Glazman I.M., 1981). Yet these interactions are exactly solvable models and their systematic study allows us to better understand their properties. Hounkonnou M. and his co-authors studied the one and N parameters models of δ'-sphere interactions called of the first and the second type (Hounkonnou M. et al., 1999). For these both models provide the basic properties and discuss the stationary scattering theory. But as indicated our previous report (Vyabandi A. and Shabani J.; 2020), the study of the $2N$ parameters models which unifies the δ-sphere interactions of the first and the second type allows us to better understand the dynamics of the perturbed physic system in term of scattering data. This is missing in the literature for the δ' potential and this is the aim of this paper. We discuss the basic properties of new exactly solvable models of $2N$ parameters models of finitely many nonrelativistic δ'-sphere and δ'-sphere plus Coulomb interactions in three space dimensions using the theory of the self-adjoint extensions of symmetric closed operators in Hilbert spaces. This model generalizes the two models introduced by Hounkonnou (Hounkonnou M. et al., 1999) and Antoine and (Antoine J.P et al., 1988) where similar works was done for 2 and N parameters of nonrelativistic δ'-sphere and δ'-sphere plus Coulomb interactions.

The paper is organized as follow. In Sec.II, we provide a mathematical definition of the hamiltonian and obtain new results on the resolvent equation, the spectral properties and the scattering data (Scattering matrix, amplitude, length and the differential scattering cross section).

In Sec.III, we generalize the results of Sec.II to the case of a two parameters δ'sphere interaction plus a Coulomb interaction.

2. Basic properties of the nonrelativistic $2N$ parameters model of finitely δ'-interaction supported by concentric spheres

2.1 Definition of the model

In this section, let us provide the mathematical definition of quantum hamiltonian describing the nonrelativistic $2N$ parameters models of $\boldsymbol{\delta'}$-sphere interaction supported by N concentric spheres of radii $0 < R_1 <\ldots< R_N$ using the theory of self adjoint extensions of symmetric closed operator.

Consider the formal expression

$$H = -\Delta + \sum_{i=1}^{N} \alpha_i \delta'(|x| - R_i),$$
$$x \in \mathbb{R}^3, 0 < R_1 <\ldots< R_N. \qquad (2.1)$$

We define in $L^2(\mathbb{R}^3)$ the closed symmetric and non negative operator by

$$\dot{H} = -\Delta,$$
$$\mathcal{D}(\dot{H}) = \{f \epsilon\, H^{2,2}(\mathbb{R}^3)/f(\partial\overline{k(0,R_\iota)}) = f'(\partial\overline{k(0,R_\iota)}) = 0\} \qquad (2.2)$$

where $H^{m,n}(\Omega)$ is the Sobolev space of indices (m,n) and $\overline{k(0,R_\iota)}$ is the closed ball of radius R centered at the origin in $\mathbb{R}^3$.

We decompose the state Hilbert space $L^2(\mathbb{R}^3)$ with respect to angular momenta by:

$$L^2(\mathbb{R}^3) = L^2((0,\infty), r^2dr) \otimes L^2(S^2). \qquad (2.3)$$

We introduce the unitary transformation by

$$U: L^2((0,\infty), r^2dr) \to L^2((0,\infty), dr) \equiv L^2((0;\infty)),$$
$$f \to (Uf)(r) = rf(r) \qquad (2.4)$$

which enables us to represent $L^2(\mathbb{R}^3)$ by

$$L^2(\mathbb{R}^3) = \oplus_{l=0}^{\infty} U^{-1}L^2((0,\infty)) \otimes [Y_l^{-l} \ldots Y_l^{-l}] \qquad (2.5)$$

where the spherical harmonics $Y_l^m, l \in \mathbb{N}_0, -l \leq m \leq l,$ provide a basis for $L^2(S^2)$ (S^2 is the unit sphere in $\mathbb{R}^3$). [...] denotes the linear span of vectors in $L^2(S^2)$.

With respect to the decomposition (2.5), $\dot{H}$ reads:

$$\dot{H} = \oplus_{l=0}^{\infty} U^{-1}\dot{h}_{l,\{R\}}U \otimes 1\mathrm{I} \qquad (2.6)$$

where

$$\dot{h}_{l,\{R\}} = -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2}$$
$$\mathcal{D}(\dot{h}_{l,\{R\}}) = \{f \in L^2((0,\infty))/f, f' \in AC_{loc}((0,\infty));$$
$$f(0+) = 0 \text{ if } l = 0; f(R_i \pm) = f'(R_i \pm) = 0,$$
$$-f'' + l(l+1)r^{-2}f \in L^2((0,\infty))\}, \qquad (2.7)$$
$$i = 1, \ldots, N, \qquad \{R\} = \{R_1, \ldots, R_N\}$$

$AC_{loc}((\Omega))$ denotes the set of locally absolutely continuous functions on $\Omega \subset \mathbb{R}$ and $f(x\pm) = \lim_{\epsilon\to 0} f(x\pm\epsilon)$. The adjoint $\dot{H}^*$ of $\dot{H}$ is given by

$$\dot{H}^* = \oplus_{l=0}^{\infty} U^{-1}\dot{h}_{l,\{R\}}^* U \otimes 1\mathrm{I} \qquad (2.8)$$

Where the adjoint $\dot{h}_l^*$ of $\dot{h}_l$ reads:

$$\dot{h}_{l,\{R\}}^* = -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2}$$
$$\mathcal{D}(\dot{h}_{l,\{R\}}^*) = \{f \in L^2((0,\infty))/f, f' \in AC_{loc}((0,\infty) \setminus \{R\});$$
$$-f'' + l(l+1)r^{-2}f \in L^2((0,\infty))\}. \qquad (2.9)$$

The equation

$$\dot{h}^*_{l,\{R\}}\phi_l(k) = k^2\phi_l(k), \phi_l \in \mathcal{D}(\dot{h}^*_{l,\{R\}}), \mathrm{Im}\, k > 0, \quad l \in \mathbb{N}_0 \qquad (2.10)$$

has two linearly independent solutions :

$$\phi_l^{(1)}(k,r) = \begin{cases} G_l(k,R_i)F_l(k,r); & r < R_i \\ 0; & r > R_i \end{cases} \qquad (2.11)$$

$$\phi_l^{(2)}(k,r) = \begin{cases} 0; & r < R_i \\ F_l(k,R_i)G_l(k,r) & r > R_i \end{cases} \qquad (2.12)$$

where

$$F_l(k,r) = \Gamma\left(l+\frac{3}{2}\right)\left(\frac{k}{2}\right)^{-l-\frac{1}{2}} r^{\frac{1}{2}} J_{l+\frac{1}{2}}(kr)$$
$$G_l(k,r) = \mathrm{i}\frac{\pi}{2}\Gamma\left(l+\frac{3}{2}\right)\left(\frac{k}{2}\right)^{-l-\frac{1}{2}} r^{\frac{1}{2}} H^{(1)}_{l+\frac{1}{2}}(kr) \qquad (2.13)$$

and $J_\nu(.)$ denote the Bessel function and $H_\nu^{(1)}(.)$ the Hankel function of the first type of order ν. Therefore, $\dot{h}_{l,\{R\}}$ has defiency indices $(2N, 2N)$ and consequently, all self-adjoint(s.a) extensions of $\dot{h}_{l,\{R\}}$ are given by a $4N^2$-parameters family of self adjoint operators (Akhiezer W.I. and Glazman I.M., 1981).

Let us define a special $2N$-parameters family of self adjoint extensions of $\dot{h}_{l,\{R\}}$ with separated boundary conditions

$$h_{l,\alpha,\beta,\{R\}} = -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2},$$
$$\mathcal{D}(h_{l,\alpha,\beta,\{R\}}) = \left\{ \phi \left| \begin{array}{l} \left(1+\frac{\alpha_{l,i}}{2}\right)\phi'(R_i+) + \left(\frac{\alpha_{l,i}}{2}-1\right)\phi'(R_i-) = 0 \\ \left(1+\frac{\beta_{l,i}}{2}\right)\phi(R_i+) + \left(\frac{\beta_{l,i}}{2}-1\right)\phi(R_i-) = 0 \end{array} \right. \right\},$$
$$\phi \epsilon \mathcal{D}(\dot{h}^*_{l,\{R\}}), \quad \alpha_{l,i}, \beta_{l,i} \epsilon \mathbb{R}, i = 1, \dots, N \qquad (2.14)$$

The Hamiltonian $h_{l,\alpha,\beta,\{R\}}$ gives the mathematical definition of the formal expression

$$-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + \sum_{i=1}^N \alpha_{l,i}\delta'(r-R_i) \qquad (2.15)$$
$$R_i > 0, \qquad i = 1, \dots, N$$

The case $\alpha_{l,i} = \beta_{l,i} = 0$ in eq. (2.14) coincides with the free kinetic energy hamiltonian $h_{l,0}$ for a fixed angular momentum l. Let $\alpha = \{\alpha_{l,i}\}_{l\in\mathbb{N}_0}, \beta = \{\beta_{l,i}\}_{l\in\mathbb{N}_0}$ and introduce in $L^2(\mathbb{R}^3)$ the operator:

$$H_{\alpha,\beta} = \oplus_{l=0}^{\infty} U^{-1} h_{l,\alpha,\beta,\{R\}} U \otimes 1\mathrm{I}. \qquad (2.16)$$

The hamiltonian $H_{\alpha,\beta}$ provides a mathematical definition of the $2N$ parameters nonrelativistic δ'-sphere interaction.

The particular case $\alpha = \beta = 0$ in eq. (2.16) leads to the free hamiltonian

$$H_0 = -\Delta; \mathcal{D}(H_0) = H^{2,2}(\mathbb{R}^3). \qquad (2.17)$$

The particular cases $\alpha \neq 0, \beta = 0$ and $\alpha = 0, \beta \neq 0$ in eq. (2.16) yield the N parameters nonrelativistic δ'-sphere interactions of the first and second type respectively (Hounkonnou M. et al., 1999).

2.2 The resolvent equation

Theorem 2.1: The resolvent of h_{l,α_l,β_l} and $H_{\alpha,\beta}$ read respectively:

(i) $$\left(h_{l,\alpha,\beta,\{R\}} - k^2\right)^{-1} = \left(h_{l,0} - k^2\right)^{-1} + \sum_{i=1}^N [\mu_{ij}^{(1)}(\tilde{\psi}_l^{(2)}(-\bar{k}),.)\,\psi_l^{(2)}(k) - \mu_{ij}^{(2)}(\psi_l^{(2)}(-\bar{k}),.)\tilde{\psi}_l^{(2)}(k)$$
$$-\mu_{ij}^{(3)}(\psi_l^{(2)}(-\bar{k}),.)\tilde{\psi}_l^{(1)}(k) + +\mu_{ij}^{(3)}(\tilde{\psi}_l^{(2)}(-\bar{k}),.)\psi_l^{(1)}(k)],$$
$$k^2 \in \rho(h_{l,\alpha,\beta,\{R\}});\ \mathrm{Im}\, k > 0;\ l \in \mathbb{N}_0 \qquad (2.18)$$

where

$$[\mu_{ij}^{(1)}(k)]^{-1} = \begin{cases} \frac{1}{\alpha_{l,i}} - \frac{\beta_{l,i}}{4} + \frac{\beta_{l,i}}{\alpha_{l,i}}[G_l(k,R_i)F_l(k,R_i)]' - [G_l(k,R_i)F_l(k,R_i)]', & i = j \\ -[G_l(k,R_i)F_l(k,R_i)]', & i \neq j, \end{cases}$$

$$[\mu_{ij}^{(2)}(k)]^{-1} = \begin{cases} \frac{1}{\beta_{l,i}} - \frac{\alpha_{l,i}}{4} - \frac{\alpha_{l,i}}{\beta_{l,i}}[G_l(k,R_i)F_l(k,R_i)]' + \frac{1}{2}[G_l(k,R_i)F_l(k,R_i)]', & i = j \\ \frac{1}{2}[G_l(k,R_i)F_l(k,R_i)]', & i \neq j \end{cases}$$

$$[\mu_{ij}^{(3)}(k)]^{-1} = \begin{cases} \frac{1}{2}\left\{\frac{1}{\alpha_{l,i}\beta_{l,i}} - \frac{1}{4} - \frac{1}{2}\left(\frac{1}{\alpha_{l,i}} - \frac{1}{\beta_{l,i}}\right)[G_l(k,R_i)F_l(k,R_i)]'\right\}, & i = j \\ -\frac{1}{8} & i \neq j, \end{cases} \qquad (2.19)$$

$$\psi_l^{(p)}(k,r) = \begin{cases} G_l(k,R_i)F_l(k,r); & r < R_i \\ (-1)^n F_l(k,R_i)G_l(k,r), & r > R_i, p = 1,2 \end{cases} \qquad (2.20)$$

$$\tilde{\psi}_l^p(k,r) = \begin{cases} G_l'(k,R_i)F_l(k,r), & r < R_i \\ (-1)^p F_l'(k,R_i)G_l(k,r), & r > R_i, p = 1,2 \end{cases} \qquad (2.21)$$

(ii) $$\left(H_{\alpha,\beta,\{R\}} - k^2\right)^{-1} = (H_0 - k^2)^{-1} + \sum_{i=1}^N \oplus_{l=0}^{\infty}\oplus_{m=-l}^{l} [\mu_{ij}^{(1)}(k)(|.|\tilde{\psi}_l^{(2)}(-\bar{k})Y_l^{(m)},.)\mathrm{x}$$
$$|.|\psi_l^{(2)}(k)Y_l^{(m)} \text{-} \mu_{ij}^{(2)}(k)(|.|\psi_l^{(2)}(-\bar{k})Y_l^{(m)},.)|.|\tilde{\psi}_l^{(2)}(k)Y_l^{(m)} - \mu_{ij}^{(3)}(k)(|.|\psi_l^{(2)}(-\bar{k})Y_l^{(m)},.)\mathrm{x}|.|\tilde{\psi}_l^{(1)}(k)Y_l^{(m)}$$
$$+\mu_{ij}^{(3)}(k)(|.|\tilde{\psi}_l^{(2)}(-\bar{k})Y_l^{(m)},.)|.|\psi_l^{(1)}(k)Y_l^{(m)}],$$
$$k^2 \in \rho(H_{\alpha,\beta,\{R\}});\ \mathrm{Im}\, k > 0, l \in \mathbb{N}_0 \qquad (2.22)$$

Proof:

Let $g \in L^2((0;\infty))$ and define the function $f(k,r)$ by

$$f(k,r) = \left(\left(h_{l,\alpha,\beta,\{R\}} - k^2\right)^{-1} g\right)(r). \qquad (2.23)$$

Since $f \in \mathcal{D}(h_{l,\alpha,\beta,\{R\}})$, it follows that f satisfies the boundary conditions in eq.(2.14). The implementation of these boundary conditions prove Eq (18).

Let us now provide the additional information on the domain of $h_{l\alpha,\beta,\{R\}}$and prove that the two parameters δ' interaction is local.

Theorem 2.2: The domain $\mathcal{D}(h_{l,,\alpha,\beta,\{R\}}), -\infty < \alpha_{l,n}, \beta_{l,n} \leq \infty, r \in \mathbb{R}, n = 1, \dots, N$consists of functions of the type:

$$\phi_{l,,\alpha,\beta,\{R\}}(k,r) = \chi_l(k,r) + $$
$$+\sum_{i=1}^{N}\Big[\mu_{ij}^{(1)}(k)\chi_l'(k,R_i)\psi_l^{(2)}(k,r) - \mu_{ij}^{(2)}(k)\chi_l(k,R_i)\tilde{\psi}_l^{(2)}(k,r)$$
$$-\mu_{ij}^{(3)}(k)\chi_l(k,R_i)\tilde{\psi}_l^{(1)}(k,r) + \mu_{ij}^{(3)}(k)\chi_l'(k,R_i)\psi_l^{(1)}(k,r)\Big],$$
$$\chi_l \in \mathcal{D}(h_{l,0}), k^2 \in \rho(h_{l,\alpha,\beta,\{R\}}), \mathrm{Im}k > 0 \qquad (2.24)$$

where the constants $\mu_{ij}^{(p)}, p = 1, \dots, 3$ are defined by eq.(2.19). The decomposition (2.24) is unique and with $\phi_{l,\alpha,\beta,\{R\}}$ of this form, we obtain:

$$(h_{l,\alpha,\beta,\{R\}} - k^2)\phi_{l,\alpha,\beta,\{R\}} = (h_{l,0} - k^2)\chi_l. \qquad (2.25)$$

Consider $\phi_{l,\alpha,\beta,\{R\}} \in \mathcal{D}(h_{l,\alpha,\beta,\{R\}})$ and $\phi_{l,\alpha,\beta,\{R\}} = 0$ in an open set$\wedge \subset [0,\infty)$, then $h_{l,\alpha,\beta,\{R\}}\phi_{l,\alpha,\beta,\{R\}} = 0$ in $\wedge$, which means that the interaction described by $h_{l,\alpha,\beta,\{R\}}$ is local.

Proof:

The domain $\mathcal{D}(h_{l,\alpha,\beta,\{R\}})$ reads:

$$\mathcal{D}(h_{l,\alpha,\beta,\{R\}}) = (h_{l,\alpha,\beta,\{R\}} - k^2)^{-1}(h_{l,0} - k^2)\mathcal{D}(h_{l,0})$$
$$= \Big\{(h_{l,0} - k^2)^{-1} + \sum_{i=1}^{N}[\mu_{ij}^{(1)}(\tilde{\psi}_l^{(2)}(-\bar{k}),.)\,\psi_l^{(2)}(k) - \mu_{ij}^{(2)}(\psi_l^{(2)}(-\bar{k}),.)\tilde{\psi}_l^{(2)}(k)$$
$$-\mu_{ij}^{(3)}\left(\psi_l^{(2)}(-\bar{k}),.\right)\tilde{\psi}_l^{(1)}(k)$$
$$+\mu_{ij}^{(3)}\left(\tilde{\psi}_l^{(2)}(-\bar{k}),.\right)\psi_l^{(1)}(k)\Big]\Big\}(h_{l,0} - k^2)\mathcal{D}(h_{l,0}),$$
$$k^2 \in \rho(h_{l,\alpha,\beta,\{R\}});\ \mathrm{Im}\,k > 0, l \in \mathbb{N}_0. \qquad (2.26)$$

Eq.(2.26) prove eq.(2.24). Let now $\phi_{l,\alpha,\beta,\{R\}} = 0$ in eq.(2.24), then

$$\chi_l(k,r) = \sum_{i=1}^{N}\Big[-\mu_{ij}^{(1)}(k)\chi_l'(k,R_i)\psi_l^{(2)}(k,r) + \mu_{ij}^{(2)}(k)\chi_l(k,R_i)\tilde{\psi}_l^{(2)}(k,r) + \mu_{ij}^{(3)}(k)\chi_l(k,R_i)\tilde{\psi}_l^{(1)}(k,r)$$
$$- \mu_{ij}^{(3)}(k)\chi_l'(k,R_i)\psi_l^{(1)}(k,r)\Big], \qquad (2.27)$$

and $\chi_l \in AC_{loc}((0,\infty))$ implies $\chi_l = 0$ which prove the uniquess of eq.(2.24). The relation

$$(h_{l,\alpha,\beta,\{R\}} - k^2)^{-1}(h_{l,0} - k^2)\chi_l$$
$$+\sum_{i=1}^{N}\Big[\mu_{ij}^{(1)}(k)(\tilde{\psi}_l^{(2)}(-\bar{k}),(h_{l,0} - k^2)\chi_l)\psi_l^{(2)}(k)$$
$$-\mu_{ij}^{(2)}(k)(\psi_l^{(2)}(-\bar{k}),(h_{l,0} - k^2)\chi_l)\tilde{\psi}_l^{(2)}(k)$$
$$-\mu_{ij}^{(3)}(k)(\psi_l^{(2)}(-\bar{k}),(h_{l,0} - k^2)\chi_l)\tilde{\psi}_l^{(1)}(k)$$
$$+\mu_{ij}^{(3)}(k)\left(\tilde{\psi}_l^{(2)}(-\bar{k}),(h_{l,0} - k^2)\chi_l\right)\psi_l^{(1)}(k)\Big]$$
$$= \phi_{l,\alpha,\beta,\{R\}}, k^2 \in \rho(h_{l,\alpha,\beta,\{R\}}), \mathrm{Im}\,k > 0 \qquad (2.28)$$

prove (2.31).

Next prove now the interaction $h_{l,\alpha,\beta,\{R\}}$ is local.

(a) We assume $R \notin \wedge$, then

$$(h_{l,0} - k^2)\sum_{i=1}^{N}\Big\{\mu_{ij}^{(1)}(k)\chi_l'(k,R_i)\,\psi_l^{(2)}(k,.) + \mu_{ij}^{(2)}(k)\chi_l(k,R_i)\tilde{\psi}_l^{(2)}(k,.) + \mu_{ij}^{(3)}(k)\chi_l(k,R_i)\tilde{\psi}_l^{(1)}(k,.)$$
$$-\mu_{ij}^{(3)}(k)\chi_l'(k,R_i)\psi_l^{(1)}(k,.)\Big\}(r) = 0 \ (2.29)$$

Implies

$$(h_{l,\alpha,\beta,\{R\}}\phi_{l,\alpha,\beta,\{R\}})(r) = k^2\phi_{l,\alpha,\beta,\{R\}}(r) + \left((h_{l,0} - k^2)\chi_l\right)(r)$$
$$= (h_{l,0} - k^2)\sum_{i=1}^{N}\Big\{\mu_{ij}^{(1)}(k)\chi_l'(k,R_i)\,\psi_l^{(2)}(k,.) + \mu_{ij}^{(2)}(k)\chi_l(k,R_i)\tilde{\psi}_l^{(2)}(k,.) + \mu_{ij}^{(3)}(k)\chi_l(k,R_i)\tilde{\psi}_l^{(1)}(k,.)$$
$$-\mu_{ij}^{(3)}(k)\chi_l'(k,R_i)\psi_l^{(1)}(k,.)\Big\}(r) = 0, r \in \wedge \qquad (2.30)$$

(b) If $R \in \wedge$ then $\phi_{l,\alpha,\beta,\{R\}}(R) = 0$ and $\chi_l \in AC_{loc}((0;\infty))$ implies χ_l= 0 and hence

$$(h_{l,,\alpha,\beta,\{R\}}\phi_{l,,\alpha,\beta,\{R\}})(r) = k^2\phi_{l,,\alpha,\beta,\{R\}}(r) = 0, r \in \wedge. \ (2.31)$$

We proved that if $\phi_{l,,\alpha,\beta,\{R\}} = 0$ in a open set$\wedge \subset \mathbb{R}$, then $h_{l,,\alpha,\beta,\{R\}}\phi_{l,,\alpha,\beta,\{R\}} = 0$ in $\wedge$, which mean that the interaction $h_{l,,\alpha,\beta,\{R\}}$is local.

2.3 Spectral properties

The spectral properties of $h_{l,\alpha,\beta,\{R\}}$are provided by the following theorem where σ_{ess}; σ_{ac}; σ_{sc} denote the essential spectrum, absolutely continuous spectrum, singularly continuous spectrum respectively.

Theorem 2.3: For all $\alpha_{l,n}, \beta_{l,n} \in (0,\infty), n = 1, \dots, N$ we have :

$$\sigma_{ess}(h_{l,\alpha,\beta,\{R\}}) = \sigma_{ac}(h_{l,\alpha,\beta,\{R\}}) = [0,\infty), \qquad (2.32)$$
$$\sigma_{sc}(h_{l,\alpha_l,\beta_l}) = \emptyset. \qquad (233)$$

Proof :

Eq. (2.32) and eq.(2.33) follow from Weyl's theorem (Newton R.G., 1966), p.112 and theorem XIII respectively.

2.4 Scattering theory for the pair ($h_{l,\alpha,\beta,\{R\}}$, $h_{l,0}$)

For $k \geq 0$, we define the function

$$\phi_{l\alpha,\beta,\{R\}}(k,r) = \mathrm{F_l}(k,r) + \sum_{i=1}^{N}\Big[\mu_{ij}^{(1)}(k)F_l'(k,r)\psi_l^{(2)}(k,r) - \mu_{ij}^{(2)}(k)F_l(k,R_i)\tilde{\psi}_l^{(2)}(k,r) - \mu_{ij}^{(3)}(k)F_l(k,R_i)\tilde{\psi}_l^{(1)}(k,r)$$
$$+\mu_{ij}^{(3)}(k)F_l'(k,R_i)\psi_l^{(1)}(k,r)\Big] \qquad (2.34)$$

The phase shift of $h_{l,\alpha_l,\beta_l}$is obtained through the asymptotic behavior of $h_{l,\alpha,\beta,\{R\}}$as $r \longrightarrow \infty$.

For $k > 0$, one has (Abramowitz M. and Stegurn I.A., 1972)

$$\phi_{l,\alpha,\beta,\{R\}}\overrightarrow{r \longrightarrow \infty}\, A_l(k)\sin\left(kr - \frac{l\pi}{2}\right) +$$

$$\sum_{i=1}^{N}\left\{\mu_{ij}^{(1)}(k)F_l'(k,R_i)F_l(k,R_i)B_l(k)\exp\left[-\mathrm{i}\left(kr-\frac{l\pi}{2}\right)\right]\right.$$
$$\left.\mu_{ij}^{(2)}(k)F_l'(k,R_i)F_l(k,R_i)B_l(k)\exp\left[-\mathrm{i}\left(kr-\frac{l\pi}{2}\right)\right]\right\}$$
$$=[A_l(k)-\mathrm{i}B_l(k)\sum_{i=1}^{N}\left(\mu_{ij}^{(1)}(k)-\mu_{ij}^{(2)}(k)\right)\times$$
$$F_l'(k,R_i)F_l(k,R_i)]\sin\left(kr-\frac{l\pi}{2}\right)$$
$$-B_l(k)\sum_{i=1}^{N}\left(\mu_{ij}^{(1)}(k)-\mu_{ij}^{(2)}(k)\right)F_l'(k,R_i)F_l(k,R_i)\times$$
$$\cos\left(kr-\frac{l\pi}{2}\right)$$
$$=\left[L_{1,\{R\}}^2(k)+L_{2,\{R\}}^2(k)\right]^{\frac{1}{2}}\sin\left(kr-\frac{l\pi}{2}+\delta_{l,\alpha,\beta,\{R\}}(k)\right)$$
$$+0(1). \qquad (2.35)$$

Therefore the phase shifts reads:

$$\delta_{l,\alpha,\beta,\{R\}}(k)=-\arctan\frac{L_{2,\{R\}}}{L_{1,\{R\}}}$$
$$=-\arctan\frac{B_l(k)\sum_{i,j=1}^{N}\Omega_{ij}(k)F_l'(k,R_i)F_l(k,R_i)}{A_l(k)-\mathrm{i}B_l(k)\sum_{,ji=1}^{N}\Omega_{ij}(k)F_l'(k,R_i)F_l(k,R_i)} \qquad (2.36)$$

where (Vyabandi A. and Shabani J.; 2020)

$$A_l(k)=2^{-l}k^{-l-1}\Gamma(2l+2)\Gamma(l+1)^{-1}, \qquad (2.37)$$
$$B_l(k)=\frac{1}{k\,A_l(k)} \qquad (2.38)$$

and

$$\Omega_{ij}(k)=\left(\mu_{ij}^{(1)}(k)-\mu_{ij}^{(2)}(k)\right).$$

The on-shell scattering matrix is given by

$$S_{l,\alpha,\beta,\{R\}}(k)=\exp(2\mathrm{i}\delta_{l,\alpha,\beta,\{R\}}(k))$$
$$=1-2\mathrm{i}kB_l^2\sum_{i,j=1}^{N}\Omega_{ij}(k)F_l'(k,R_i)F_l(k,R_i) \quad (2.39)$$

The partial wave scattering amplitude is given by:

$$f_{l,\alpha,\beta,\{R\}}(k)=\frac{\exp\left(2\mathrm{i}\delta_{l,\alpha,\beta,\{R\}}(k)\right)-1}{2\mathrm{i}k}$$
$$=-B_l^2(k)\sum_{i,j=1}^{N}\Omega_{ij}(k)F_l'(k,R_i)F_l(k,R_i). \quad (240)$$

The on-shell scattering amplitude $f_{l,\alpha,\beta,\{R\}}(k,\omega,\omega')$ associated with $H_{\alpha,\beta}$ reads:

$$f_{l\alpha,\beta,\{R\}}(k,\omega,\omega')$$
$$=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{\exp\left(2\mathrm{i}\delta_{l,\alpha,\beta,\{R\}}(k)\right)-1}{2\mathrm{i}k}\overline{Y_l^m(\omega')}Y_l^m(\omega). \quad (2.41)$$

The corresponding effective range expansion reads:

$$[(2l+1)!!]^2k^{2l+1}\cot\delta_{l,\alpha,\beta,\{R\}}(k)$$
$$=-a_{l,\alpha,\beta,\{R\}}^{-1}+\frac{1}{2}r_{l,\alpha,\beta,\{R\}}k^2$$
$$+0(k^4) \quad (2.42)$$

where the scattering length $a_{l\alpha,\beta,\{R\}}$ is given by:

$$a_{l,\alpha,\beta,\{R\}}=\sum_{i,j=1}^{N}\Omega_{ij}(0)\,(l+1)R_i^{2l+1}. \qquad (2.43)$$

The total scattering section for the pair ($h_{l,\alpha,\beta,\{R\}}$,$h_{l,0}$) is given by

$$\sigma_{total}=\int d\Omega\sigma(\theta,\varphi) \qquad (2.44)$$

where θ,φ are angulaire variables. The differential scattering cross section σ reads (Newton R.G., 1966)

$$\sigma=4\sum_{l=0}^{\infty}\left(l+\frac{1}{2}\right)^2\left|f_{l,\alpha,\beta,\{R\}}\right|^2P_l^2(\cos\theta). \qquad (2.45)$$

The Legendre Polynomials $P_l(\cos\theta)$ are defined by Rodrigues formula (Vyabandi A. and Shabani J.; 2020):

$$P_l(\cos\theta)=\frac{(-)^l}{2^l l!}\left(\frac{\mathrm{d}}{\mathrm{d}\cos\theta}\right)(\sin\theta)^{2l}. \qquad (2.46)$$

Then eq. (53) reads:

$$\sigma_{total}=4\sum_{l=0}^{\infty}\left(l+\frac{1}{2}\right)\left|f_{l,\alpha,\beta,\{R\}}\right|^2\int\mathrm{d}\Omega\left(l+\frac{1}{2}\right)P_l(\cos\theta)\,P_l(\cos\theta)$$
$$=4\sum_{l=0}^{\infty}\left(l+\frac{1}{2}\right)\left|f_{l,\alpha,\beta,\{R\}}\right|^2\int_0^{2\pi}d\varphi\int_0^{\pi}\mathrm{d}\theta\sin\theta\left(l+\frac{1}{2}\right)P_l(\cos\theta)P_l(\cos\theta)$$
$$=-8\pi\sum_{l=0}^{\infty}\left(l+\frac{1}{2}\right)\left|f_{l,\alpha,\beta,\{R\}}\right|^2\int_0^{\pi}\mathrm{d}(\cos\theta)\left(l+\frac{1}{2}\right)P_l(\cos\theta)P_l(\cos\theta)$$
$$=8\pi\sum_{l=0}^{\infty}\left(l+\frac{1}{2}\right)\left|f_{l,\alpha,\beta,\{R\}}\right|^2\int_{-1}^{1}\mathrm{d}x\left(l+\frac{1}{2}\right)P_l(x)P_l(x) \quad (2.47)$$

with x = $\cos\theta$, then, the total scattering cross section reads :

$$\sigma_{total}=8\pi\sum_{l=0}^{\infty}\left(l+\frac{1}{2}\right)\left|f_{l,\alpha,\beta,\{R\}}\right|^2$$
$$=8\pi\sum_{l=0}^{\infty}\left(l+\frac{1}{2}\right)\left|B_l^2(k)\sum_{i,j=1}^{N}\Omega_{ij}(k)F_l'(k,R_i)F_l(k,R_i)\right|^2. \quad (2.48)$$

A straightforward computation shows that:

$$\sigma_{total}=\sum_{l=0}^{\infty}\sigma_l \qquad (2.49)$$

where σ_l is called partial cross sections and is given by :

$$\sigma_l=4\pi(2l+1)k^{-2}\sin^2\delta_{l,\alpha,\beta,\{R\}}.$$

3. Basic properties of the two-parameters nonrelativistic δ'-sphere interaction plus Coulomb interaction

3.1 Definition of the model

The $2N$-parameters $\boldsymbol{\delta'}$-sphere interaction plus Coulomb interaction is formally given by the hamiltonian

$$H=-\Delta+\frac{\gamma}{|x|}+\sum_{i=1}^{N}\alpha_{l,i}\delta'(|x|-R_i),\quad \gamma\in\mathbb{R},\quad x\in\mathbb{R}^3,$$
$$0<R_1<R_2<\cdots<R_N. \qquad (3.1)$$

Consider in $L^2(\mathbb{R}^3)$ the operator

$$\dot{H}_{\gamma,\{R\}}=-\Delta+\frac{\gamma}{|x|},$$

$\mathcal{D}(\dot{H}_{\gamma,\{R\}}) = \{f\epsilon\, H^{2,2}(\mathbb{R}^3)/f(\partial\overline{k(0,R_\iota)}) = f'(\partial\overline{k(0,R_\iota)}) = 0\}$. (3.2)

We introduce the operator $\dot{H}_\gamma$ by

$$\dot{H}_{\gamma,\{R\}} = \oplus_{l=0}^{\infty} U^{-1}\dot{h}_{l,\gamma,\{R\}} U \otimes 1\mathrm{I} \quad (3.3)$$

where

$$\dot{h}_{l,\gamma,\{R\}} = -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + \frac{\gamma}{r}$$
$$\mathcal{D}(\dot{h}_{l,\gamma,\{R\}}) = \{f \in L^2((0,\infty))/f, f' \in AC_{loc}((0,\infty)); f(0+) = 0 \text{ if } l = 0; f(R_i \pm) = f'(R_i \pm) = 0, -f'' + l(l+1)r^{-2}f + \gamma r^{-1}f \in L^2((0,\infty))\}, \gamma \in \mathbb{R}, l \in \mathbb{N}_0, i = 1,2,\dots,N, \{R\} = \{R_1, R_2, \dots, R_N\} \quad (3.4)$$

The adjoint $\dot{H}^*_{\gamma,\{R\}}$ of $\dot{H}_{\gamma,\{R\}}$ reads

$$\dot{H}^*_{\gamma,\{R\}} = \oplus_{l=0}^{\infty} U^{-1}\dot{h}^*_{l,\gamma,\{R\}} U \otimes 1\mathrm{I} \quad (3.5)$$

where the adjoint $\dot{h}^*_{l,\gamma,\{R\}}$ of $\dot{h}_{l,\gamma,\{R\}}$ reads:

$$\dot{h}^*_{l,\gamma,\{R\}} = -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + \frac{\gamma}{r}$$
$$\mathcal{D}(\dot{h}^*_{l,\gamma,\{R\}}) = \{f \in L^2((0,\infty))/f, f' \in AC_{loc}((0,\infty) \setminus \{R\}); f(0+) = 0 \text{ if } l = 0; -f'' + l(l+1)r^{-2}f + \gamma r^{-1}f \in L^2((0,\infty))\}, \gamma \in \mathbb{R}, l \in \mathbb{N}_0. \quad (3.6)$$

The deficiency indices equation

$$\dot{h}^*_{l,\gamma,\{R\}}\phi_{l,\gamma}(k) = k^2\phi_{l,\gamma}(k), \phi_{l,\gamma} \in \mathcal{D}(\dot{h}^*_{l,\gamma,\{R\}}), \mathrm{Im}\, k > 0, l \in \mathbb{N}_0 \quad (3.7)$$

has $2N$ linearly independent solutions :

$$\phi^{(1)}_{l,\gamma,i}(k,r) = \begin{cases} G_{l,\gamma}(k,R_i)F_{l,\gamma}(k,r); & r < R_i \\ 0 & r > R_i \end{cases} \quad (3.8)$$

$$\phi^{(2)}_{l,\gamma,i}(k,r) = \begin{cases} 0; & r < R_i \\ F_{l,\gamma}(k,R_i)G_{l,\gamma}(k,r) & r > R_i \end{cases} \quad (3.9)$$

Where $i = 1, \dots, N$ and

$$F_{l,\gamma}(k,r) = r^{l+1}\exp(ikr)\,{}_1F_1\left(l+1+\frac{i\gamma}{2k}; 2l+2; -2ikr\right)$$
$$G_{l,\gamma}(k,r) = \Gamma(2l+2)^{-1}\Gamma\left(l+1+\frac{i\gamma}{2k}\right)(-2ik)^{2l+1} \times r^{l+1}\exp(ikr)U\left(l+1+\frac{i\gamma}{2k}; 2l+2; 2ikr\right) \quad (3.10)$$

where ${}_1F_1(a;b;r)(U(a;b;r))$ denote respectively the regular (irregular)confluent hypergeometric functions (Bolle D. and Gesztesy F., 1984). The operator $\dot{h}_{l,\gamma}$has deficiency indices $(2N, 2N)$ and consequently all its self-adjoint extensions may be parameterized by a $4N^2$- parameters family of self-adjoint operators (Akhiezer W.I. and Glazman I.M., 1981). Let us introduce the following $2N$ parameters family of s.a extensions of $\dot{h}_{l,\gamma}$by:

$$h_{l,\gamma,\alpha,\beta,\{R\}} = -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + \frac{\gamma}{r},$$
$$\mathcal{D}(h_{l,\gamma,\alpha,\beta,\{R\}}) = \left\{ f \,\middle|\, \begin{array}{l} \left(1+\frac{\alpha_{l,i}}{2}\right)f'(R_i+) + \left(\frac{\alpha_{l,i}}{2}-1\right)f'(R_i-) = 0 \\ \left(1+\frac{\beta_{l,i}}{2}\right)f(R_i+) + \left(\frac{\beta_{l,i}}{2}-1\right)f(R_i-) = 0. \end{array} \right\},$$
$$f \in \mathcal{D}(\dot{h}^*_{l,\gamma,\{R\}}), \alpha_{l,i}, \beta_{l,i} \in \mathbb{R}, i = 1,\dots,N \quad (3.11)$$

The case $\alpha_{l,i} = \beta_{l,i} = 0$ in eq. (3.11) yields the free Coulomb hamiltonian $h_{l,\gamma}$ for a fixed angular momentum l.

The $2N$-parameters nonrelativistic δ'-sphere interaction plus Coulomb interaction is defined by

$$H_{\gamma,\alpha,\beta,\{R\}} = \oplus_{l=0}^{\infty} U^{-1}h_{l,\gamma,\alpha,\beta,\{R\}} U \otimes 1\mathrm{I},$$
$$\alpha = \{\alpha_{l,i}\}_{l=0}^{\infty}, \beta = \{\beta_{l,i}\}_{l=0}^{\infty}. \quad (3.12)$$

The particular case $\alpha = \beta = 0$ in eq. (3.12) leads to the Coulomb hamiltonian

$$H_{\gamma,\{R\}} = \oplus_{l=0}^{\infty} U^{-1}h_{l,\gamma,\{R\}} U \otimes 1\mathrm{I}, \quad (3.13)$$

where

$$h_{l,\gamma,\{R\}} = -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + \frac{\gamma}{r}$$
$$\mathcal{D}(h_{l,\gamma,\{R\}}) = \{f \in L^2((0,\infty))/f, f' \in AC_{loc}((0,\infty)); -f'' + l(l+1)r^{-2}f + \gamma r^{-1}f \in L^2((0,\infty))\}, \gamma \in \mathbb{R}, l \geq 1. \quad (3.14)$$

The case $\alpha \neq 0, \beta = 0$ in eq. (3.12) yields the N parameters nonrelativistic δ'-sphere interaction plus Coulomb interaction of the first type. The case $\beta \neq 0, \alpha = 0$ in eq. (3.12) yields the N parameters nonrelativistic δ'-sphere interaction plus Coulomb interaction of the second type[1].

3.2 The resolvent equation

The resolvent of $h_{l,\gamma,\alpha,\beta,\{R\}}$and $H_{\gamma,\alpha,\beta,\{R\}}$ are given by the following theorem:

Theorem 3.1: If $\alpha_{l,n}, \beta_{l,n} \neq 0$, we have:

(i)The resolvent of $h_{l,\gamma,\alpha,\beta,\{R\}}$ is given by

$$(h_{l,\gamma,\alpha,\beta,\{R\}} - k^2)^{-1} = (h_{l,\gamma,\{R\}} - k^2)^{-1} + \sum_{i=1}^{N}[\mu^{(1)}_{ij,\gamma}(\tilde{\psi}^{(2)}_{l,\gamma}(-\bar{k}),.)\psi^{(2)}_{l,\gamma}(k) - \mu^{(2)}_{ij,\gamma}(\psi^{(2)}_{l,\gamma}(-\bar{k}),.)\tilde{\psi}^{(2)}_{l,\gamma}(k) - \mu^{(3)}_{ij,\gamma}(\psi^{(2)}_{l,\gamma}(-\bar{k}),.)\tilde{\psi}^{(1)}_{l,\gamma}(k) + +\mu^{(3)}_{ij,\gamma}(\tilde{\psi}^{(2)}_{l,\gamma}(-\bar{k}),.)\psi^{(1)}_{l,\gamma}(k)],$$
$$k^2 \in \rho(h_{l,\alpha,\beta,\{R\}});\ \mathrm{Im}\, k > 0;\ l \in \mathbb{N}_0 \quad (3.15)$$

where

$$[\mu^{(1)}_{ij,\gamma}(k)]^{-1} = \begin{cases} \left(\frac{1}{\alpha_{l,i}} - \frac{\beta_{l,i}}{4} + \frac{1}{2}\left(\frac{\beta_{l,i}}{\alpha_{l,i}} - 1\right)\right)[G_{l,\gamma}(k,R_i)F_{l,\gamma}(k,R_i)]', & i = j \\ -\frac{1}{2}[G_{l,\gamma}(k,R_i)F_{l,\gamma}(k,R_i)]', & i \neq j, \end{cases}$$

$$[\mu^{(2)}_{ij,\gamma}(k)]^{-1} = \begin{cases} \left(\frac{1}{\beta_{l,i}} - \frac{\alpha_{l,i}}{4} + \frac{1}{2}\left(1 - \frac{\alpha_{l,i}}{\beta_{l,i}}\right)\right)[G_{l,\gamma}(k,R_i)F_{l,\gamma}(k,R_i)]', & i = j \\ \frac{1}{2}[G_{l,\gamma}(k,R_i)F_{l,\gamma}(k,R_i)]', & i \neq j \end{cases}$$

$$[\mu_{ij,\gamma}^{(3)}(k)]^{-1} = \begin{cases} \left(\frac{1}{\alpha_{l,i}\beta_{l,i}} - \frac{1}{4} - \frac{1}{2}\left(\frac{1}{\alpha_{l,i}} - \frac{1}{\beta_{l,i}}\right)[G_{l,\gamma}(k,R_i)F_{l,\gamma}(k,R_i)]'\right), i = j \\ -\frac{1}{4} \qquad i \neq j, \end{cases} \quad (3.16)$$

$$\psi_{l,\gamma}^{(n)}(k,r) = \begin{cases} G_{l,\gamma}(k,R_j)F_{l,\gamma}(k,r); & r < R_j \\ (-1)^n F_{l,\gamma}(k,R_j)G_{l,\gamma}(k,r), & r > R_j, n = 1,2 \end{cases} \quad (3.17)$$

$$\tilde{\psi}_{l,\gamma}^{n}(k,r) = \begin{cases} G'_{l,\gamma}(k,R_j)F_{l,\gamma}(k,r), & r < R_j \\ (-1)^n F'_{l,\gamma}(k,R_j)G_{l,\gamma}(k,r), & r > R_j, n = 1,2 \end{cases} \quad (3.18)$$

(ii) $\left(H_{\gamma,\alpha,\beta,\{R\}} - k^2\right)^{-1} = \left(H_{\gamma,\{R\}} - k^2\right)^{-1}$

$$+ \sum_{i,j=1}^{N} \oplus_{l=0}^{\infty} \oplus_{m=-l}^{l} [\mu_{ij,\gamma}^{(1)}(k)(|.|\tilde{\psi}_{l,\gamma}^{(2)}(-\bar{k})Y_l^{(m)},.) \mathrm{x} \, |.|\psi_{l,\gamma}^{(2)}(k)Y_l^{(m)} - \mu_{ij,\gamma}^{(2)}(k)(|.|\psi_{l,\gamma}^{(2)}(-\bar{k})Y_l^{(m)},.)|.|\tilde{\psi}_{l,\gamma}^{(2)}(k)Y_l^{(m)} - \mu_{ij,\gamma}^{(3)}(k)(|.|\psi_{l,\gamma}^{(2)}(-\bar{k})Y_l^{(m)},.)\mathrm{x}|.|\tilde{\psi}_{l,\gamma}^{(1)}(k)Y_l^{(m)} + \mu_{ij,\gamma}^{(3)}(k)(|.|\tilde{\psi}_{l,\gamma}^{(2)}(-\bar{k})Y_l^{(m)},.)|.|\psi_{l,\gamma}^{(1)}(k)Y_l^{(m)}],$$

$k^2 \in \rho(H_{\gamma,\alpha,\beta,\{R\}})$; Im $k > 0, l \in \mathbb{N}_0$ (3.19)

proof :
Similar to the proof of theorem 2.1.

Let us now provide the additional information on the domain of $h_{l,\gamma,\alpha,\beta,\{R\}}$ and prove that the $2N$ parameters δ' interaction plus Coulomb interaction is local.

Theorem 3.2: The domain $\mathcal{D}(h_{l,\gamma,\alpha,\beta,\{R\}}), -\infty < \alpha_{l,n}, \beta_{l,n} \leq \infty, n = 1, \dots, N$ consists of functions of the type:

$$\phi_{l,\gamma,\alpha,\beta,\{R\}}(k,r) = \chi_{l,\gamma}(k,r) + \sum_{i=1}^{N}[\mu_{ij,\gamma}^{(1)}(k)\chi'_{l,\gamma}(k,R_i)\psi_{l,\gamma}^{(2)}(k,r) - \mu_{ij,\gamma}^{(2)}(k)\chi_{l,\gamma}(k,R_i)\tilde{\psi}_{l,\gamma}^{(2)}(k,r) - \mu_{ij,\gamma}^{(3)}(k)\chi_{l,\gamma}(k,R_i)\tilde{\psi}_{l,\gamma}^{(1)}(k,r) + \mu_{ij,\gamma}^{(3)}(k)\chi'_{l,\gamma}(k,R_i)\psi_{l,\gamma}^{(1)}(k,r)],$$

$\chi_{l,\gamma} \in \mathcal{D}(h_{l,\gamma,\{R\}}), k^2 \in \rho(h_{l,\alpha,\beta,\{R\}})$, Im$k > 0$ (3.20)

Where the constants $\mu_{ij,\gamma}^{(p)}(k), p = 1,2,3$ are defined by eq.(3.16), $\chi_{l,\gamma} \in \mathcal{D}(h_{l,\gamma,\{R\}})$ and $k^2 \in \rho(h_{l,\gamma,\alpha,\beta,\{R\}})$, Im$k > 0$. The decomposition (3.20) is unique and with $\phi_{l,\gamma,\alpha,\beta,\{R\}}$ of this form, we obtain:

$$(h_{l,\gamma,\alpha,\beta,\{R\}} - k^2)\phi_{l,\gamma,\alpha,\beta,\{R\}} = (h_{l,\gamma,\{R\}} - k^2)\chi_{l,\gamma}. \quad (3.21)$$

Consider $\phi_{l,\gamma,\alpha,\beta,\{R\}} \in \mathcal{D}(h_{l,\gamma,\alpha,\beta,\{R\}})$ and $\phi_{l,\gamma,\alpha,\beta,\{R\}} = 0$ in a open set $\Lambda \subset (0,\infty)$, then $h_{l,\gamma,\alpha,\beta,\{R\}}\phi_{l,\gamma,\alpha,\beta,\{R\}} = 0$ in Λ, which means that the interaction described by $h_{l,\gamma,\alpha,\beta,\{R\}}$ is local.

Proof
Similar to the proof of theorem 2.2.

3.3 Spectral properties of $h_{l,\gamma,\alpha,\beta,\{R\}}$

The spectral properties of $h_{l,\gamma,\alpha,\beta,\{R\}}$ are provided by the following theorem where $\sigma_{ess}, \sigma_{ac}, \sigma_{sc}$ denote the essential spectrum, absolutely continuous spectrum, singularly continuous spectrum respectively.

Theorem 3.3: For all $\alpha_{l,n}, \beta_{l,n} \in (0,\infty)$, we have:

$$\sigma_{ess}(h_{l,\gamma,\alpha,\beta,\{R\}}) = \sigma_{ac}(h_{l,\gamma,\alpha,\beta,\{R\}}) = [0,\infty) \quad (3.22)$$

$$\sigma_{sc}(h_{l,\gamma,\alpha,\beta,\{R\}}) = \emptyset. \quad (3.23)$$

Proof:
Similar to the proof of theorem 2.3.

3.4 Scattering theory for the pair ($h_{l,\gamma,\alpha,\beta,\{R\}}$, $h_{l,\gamma,\{R\}}$)

For $k \geq 0$, we define the function

$$\phi_{l,\alpha,\beta,\{R\}}(k,r) = F_{l,\gamma}(k,r) + \sum_{i=1}^{N}[\mu_{ij,\gamma}^{(1)}(k)F'_{l,\gamma}(k,r)\psi_{l,\gamma}^{(2)}(k,r) - \mu_{ij,\gamma}^{(2)}(k)F_{l,\gamma}(k,R_i)\tilde{\psi}_{l,\gamma}^{(2)}(k,r) - \mu_{ij,\gamma}^{(3)}(k)F_{l,\gamma}(k,R_i)\tilde{\psi}_{l,\gamma}^{(1)}(k,r) + \mu_{ij,\gamma}^{(3)}(k)F'_{l,\gamma}(k,R_i)\psi_{l,\gamma}^{(1)}(k,r)]. \quad (3.24)$$

One can show easily that the function $h_{l,\gamma,\alpha,\beta,\{R\}}$ fulfills the following conditions:

$$\left(1 + \frac{\alpha_{l,n}}{2}\right)\phi'_{l,\gamma,\alpha,\beta,\{R\}}(R_i+) + \left(\frac{\alpha_{l,n}}{2} - 1\right)\phi'_{l,\gamma,\alpha,\beta,\{R\}}(R_i-) = 0$$

$$\left(1 + \frac{\beta_{l,n}}{2}\right)\phi_{l,\gamma,\alpha,\beta,\{R\}}(R_i+) + \left(\frac{\beta_{l,n}}{2} - 1\right)\phi_{l,\gamma,\alpha,\beta,\{R\}}(R_i-) = 0,$$

$$\phi''_{l,\gamma,\alpha,\beta,\{R\}}(k,r) + l(l+1)r^{-2}\phi_{l,\gamma,\alpha,\beta,\{R\}}(k,r) + \gamma r^{-1}\phi_{l,\gamma,\alpha,\beta,\{R\}}(k,r) = k^2\phi_{l,\gamma,\alpha,\beta,\{R\}}(k,r). \quad (3.25)$$

Consequently, the functions $\phi_{l,\gamma,\alpha,\beta,\{R\}}$ are the scattering wave functions of $h_{l,\gamma,\alpha,\beta,\{R\}}$.

For $k > 0$, the study of the asymptotic behavior of $\phi_{l,\gamma,\alpha,\beta,\{R\}}$ as $r \to \infty$ yields(Gesztesy F. and Thaller B., 1981)

$$\phi_{l,\gamma,\alpha_l\beta_l} \underset{r\to\infty}{\longrightarrow} A_{l,\gamma}(k)\sin\left(kr - \frac{\gamma}{2k}\ln(2kr) - \frac{l\pi}{2} + \delta_{l,\gamma}^{(0)}(k)\right) + \sum_{i=1}^{N}\left(\mu_{ij,\gamma}^{(1)}(k) - \mu_{ij,\gamma}^{(2)}(k)\right)F'_{l,\gamma}(k,R_i)F_{l,\gamma}(k,R_i)B_{l,\gamma}(k) \times \exp\left[-\mathrm{i}\left(kr - \frac{\gamma}{2k}\ln(2kr) - \frac{l\pi}{2} + \delta_{l,\gamma}^{(0)}(k)\right)\right]$$

$$= \left[A_{l,\gamma}(k) - \mathrm{i}B_{l,\gamma}(k)F'_{l,\gamma}(k,R_i)F_{l,\gamma}(k,R_i) \times \left(\mu_{ij,\gamma}^{(1)}(k) - \mu_{ij,\gamma}^{(2)}(k)\right)\right]\sin x_1 + B_{l,\gamma}(k)F'_{l,\gamma}(k,R_i)F_{l,\gamma}(k,R_i)\left(\mu_{ij,\gamma}^{(1)}(k) - \mu_{ij,\gamma}^{(2)}(k)\right)\cos x_1$$

$$= [h_{l,\gamma,1}^2(k) + h_{l,\gamma,2}^2(k)]^{\frac{1}{2}}\sin\left(x_1 + \delta_{l,\gamma,\alpha,\beta,\{R\}}^{(c)}(k)\right) + 0(1). \quad (3.26)$$

Where the pure Coulomb phase shift $\delta_{l,\gamma}^{(0)}(k)$ is given by

$$\delta_{l,\gamma}^{(0)}(k) = \arg\left[\Gamma\left(l + 1 + \frac{\mathrm{i}\gamma}{2k}\right)\right] \quad (3.27)$$

and

$$x_1 = kr - \frac{\gamma}{2k}\ln(2kr) - \frac{l\pi}{2} + \delta_{l,\gamma}^{(0)}(k) \quad (3.28)$$

$$h_{l,\gamma,2} = \sum_{i=1}^{N} B_{l,\gamma}(k)F'_{l,\gamma}(k,R_i)F_{l,\gamma}(k,R_i)\left(\mu^{(1)}_{ij,\gamma}(k) - \mu^{(2)}_{ij,\gamma}(k)\right) \quad (3.29)$$

$$h_{l,\gamma,1} = A_{l,\gamma}(k) - \mathrm{i}\sum_{i=1}^{N} B_{l,\gamma}(k)F'_{l,\gamma}(k,R_i)F_{l,\gamma}(k,R_i) \times \left(\mu^{(1)}_{ij,\gamma}(k) - \mu^{(2)}_{ij,\gamma}(k)\right) \quad (3.30)$$

The Coulomb modified phase shift $\delta^{(c)}_{l,\gamma}(k)$ is given by:

$$\delta^{(c)}_{l,\gamma,\alpha,\beta,\{R\}}(k) = -\arctan\frac{h_{l,\gamma,2}(k)}{h_{l,\gamma,1}(k)} = -\arctan\frac{\sum_{i=1}^{N} B_{l,\gamma}(k)F'_{l,\gamma}(k,R_i)F_{l,\gamma}(k,R_i)\Omega(k)}{A_{l,\gamma}(k) - \mathrm{i}\sum_{i=1}^{N} B_{l,\gamma}(k)F'_{l,\gamma}(k,R_i)F_{l,\gamma}(k,R_i)Y(k)} \quad (3.31)$$

where(Bolle D. and Gesztesy F., 1984)

$$A_{l,\gamma}(k) = 2^{-l}k^{-l-1}\exp\left(\frac{\pi\gamma}{4k}\right) \times \Gamma(2l+2)\left|\Gamma\left(l+1+\frac{\mathrm{i}\gamma}{2k}\right)\right|^{-1} \quad (3.33)$$

$$B_{l,\gamma}(k) = \frac{1}{k\,A_{l,\gamma}(k)} \quad (3.32)$$

and

$$\Omega_{ij,\gamma}(k) = \left(\mu^{(1)}_{ij,\gamma}(k) - \mu^{(2)}_{ij,\gamma}(k)\right)$$

The Coulomb modified on-shell scattering matrix is given by:

$$S^{(c)}_{l,\gamma,\alpha,\beta,\{R\}}(k) = \exp\left(2\mathrm{i}\delta^{(c)}_{l,\gamma,\alpha,\beta,\{R\}}(k)\right) = 1 - 2\mathrm{i}kB^2_{l,\gamma}(k)\sum_{i=1}^{N}\Omega_{ij,\gamma}(k)F'_{l,\gamma}(k,R)F_{l,\gamma}(k,R). \quad (3.33)$$

The corresponding partial wave scattering amplitude reads:

$$f^{(c)}_{l,\gamma,\alpha,\beta,\{R\}}(k) = \frac{\exp\left(2\mathrm{i}\delta^{(c)}_{l,\gamma,\alpha,\beta,\{R\}}(k)\right) - 1}{2\mathrm{i}k} = -B^2_{l,\gamma}\sum_{i=1}^{N}F'_{l,\gamma}(k,R_i)F_{l,\gamma}(k,R_i)\Omega_{ij,\gamma}(k). \quad (3.34)$$

The Coulomb modified effective range expansion corresponding to $h_{l,\gamma,\alpha,\beta,\{R\}}$reads (Abramowitz M. and Stegurn I.A., 1972)

$$(2k)^{2l}\Gamma(2l+2)^{-2}\left|\Gamma\left(l+1+\frac{\mathrm{i}\gamma}{2k}\right)\right|^2 \times \exp\left(-\frac{\pi\gamma}{2k}\right)\left(k\cot\delta^{(c)}_{l,\gamma,\alpha,\beta,\{R\}}(k) - \mathrm{i}k + \exp\left(\frac{\pi\gamma}{2k}\right)h_\gamma(k)\right) = -\frac{1}{a^{(c)}_{l,\gamma,\alpha,\beta,\{R\}}(k)} + 0(k^2), k > 0, \gamma \in \mathbb{R}. (3.35)$$

where $a^{(c)}_{l,\gamma,\alpha_l,\beta_l}(k)$ is the Coulomb modified scattering length and the function $h_\gamma(k)$ is defined by

$$h_\gamma(k) = \gamma\left|\Gamma\left(1+\frac{\mathrm{i}\gamma}{2k}\right)\right|^2\left[\frac{\mathrm{i}k}{\gamma} + \ln\left(\frac{2\mathrm{k}}{\mathrm{i}|\gamma|}\right) + \Psi\left(1+\frac{\mathrm{i}\gamma}{2k}\right)\right]$$

Ψ denotes a digamma function(Abramowitz M. and Stegurn I.A., 1972).

The Coulomb modified scattering length $a_{l,\gamma,\alpha,\beta,\{R\}}$can be calculated from the expansion (3.37). After a straightforward computation we obtained:

$$\frac{1}{a^{(c)}_{l,\gamma,\alpha,\beta,\{R\}}(k)} = \sum_{i,j=1}^{N}\frac{\Omega_{ij,\gamma}(0)^{-1}}{F'_{l,\gamma}(0,R_i)F_{l,\gamma}(0,R_i)} \quad (3.36)$$

where we have used the notations $\nu = 2l+1, y = (4\gamma r)^{\frac{1}{2}}$ and, $z = (4|\gamma|r)^{\frac{1}{2}}$.

The corresponding differential scattering cross section is

$$\sigma = 4\sum_{l=0}^{\infty}\left(l+\frac{1}{2}\right)^2\left|f^{(c)}_{l,\gamma,\alpha,\beta,\{R\}}(k)\right|^2 \quad (3.37)$$

A straightforward computation shows that the total scattering cross section for the pair $(h_{l,\gamma,\alpha,\beta,\{R\}}, h_{l,\gamma,\{R\}})$ reads:

$$\sigma_{total} = 8\pi\sum_{l=0}^{\infty}\left(l+\frac{1}{2}\right)\left|f^{(c)}_{l,\gamma,\alpha,\beta,\{R\}}(k)\right|^2 = 8\pi\sum_{l=0}^{\infty}\left(l+\frac{1}{2}\right) \times \left|-B^2_{l,\gamma}(k)\sum_{i,j=1}^{N}F'_{l,\gamma}(k,R_i)F_{l,\gamma}(k,R_i)\,\Omega_{ij,\gamma}(k)\right|^2 \quad (3.38)$$

4. Conclusion

A study of the 2N-parameter model of δ'- nonrelativistic spherical and δ' - nonrelativistic spherical interactions plus a Coulomb interaction was made with the aim of to determine the impact of the unification of two nonrelativistic δ'-sphere interactions of the first and second kind on the motion of a nonrelativistic physical system.

The scattering wave functions depend on

- N parameters which characterize the δ'- sphere interaction of the first kind
- N parameters which characterize the δ'- sphere interaction of the second kind
- the coupling of the two kinds of parameters mentioned above which characterize the unification of the δ'- sphere interactions of the first and the second kind.

The case $\alpha_{l,n} > 0, \beta_{l,n} < 0, n = 1, \dots, N$ leads to a strong increase in the diffusion amplitude and a decrease in the diffusion length.

Finally if we consider the case $\alpha_{l,n} \neq 0, \beta_{l,n} = 0, n = 1, \dots, N$ the scattering quantities theory found in the case of the $2N$ parameters model lead to those corresponding to the δ'- sphere interaction of the first kind while the case $\beta_{l,n} \neq 0, \alpha_{l,n} = 0, n = 1, \dots, N$ allows to have those corresponding to the δ'- sphere interaction of the second kind.

Bibliography

[1] Hounkonnou M.N., Hounkpe M., Shabani J. (1999), Exactly solvable models of δ'-sphere interactions in nonrelativistic quantum mechanics, *J.Math.Phys.*40,

4254-4273

[2] Antoine J.P., Gesztesy F. and Shabani J. (1988), Exactly solvable models of sphere interactions in quantum mechanics, *J.Phys.A* 20, 3687-3712

[3] Shabani J. (1988), Finitely many δ-interactions with supports on concentric sphere, *J.Math.Phys*.29, 660-664

[4] Dabrowski L. and Shabani J. (1988), finitely many sphere interactions in quantum mechanics nonseparated boundary conditions, *J.Math. Phys*.29, 2241-2244

[5] Antoine J.P., Exner P., Seba P. and Shabani J. (1994), A mathematical model of heavy quarkonia mesonic decays, *Ann. Phys. (N.Y)* 233,1-16.

[6] Shabani J. (1988), Some properties of the hamiltonian describing a finite number of δ'-interactions with supports on centric spheres, *Nuovo Cimento B*101, 429-439

[7] Hounkonnou M.N., Hounkpe M. and Shabani J. (1997), Scattering theory for finitely many sphere interactions supported by concentric spheres, *J. Math. Phys*.38, 2832-2850

[8] Dittrich J., Exener P. and Sěba P. (1989), Dirac operators with a spherically symmetric $\delta -$shell interaction, *J. Math. Phys.30*, 2875-2882

[9] Dittrich J., Exner P. and Sěba, Dirac Hamiltonian with Coulomb potential and spherically symmetric shell contact interaction, *J. Math. Phys*. **33**, 2207-2214(1992)

[10] Shabani J. and Vyabandi A. (2002), Exactly solvable models of relativistic δ -sphere interactions, *J. Math. Phys*.43, 6064-6084

[11] Green I.M. and Moszkowski S.A. (1965), Nuclear coupling schemes with a surface delta interaction, *Phys. Rev* 139 B, 790

[12] Blinder S.M. (1978), Modified delta-function potential for hyperfine interactions, *Phys. Rev.A*18, 853

[13] Rubio J. and Garcia- Moliner F. (1967), Formal theory of equivalent potentials in solidsII. Scattering theory approach for muffin-tin potentials, *Proc. Phys. Soc* 92, 206

[14] Akhiezer W.I. and Glazman I.M. (1981), Theory of linear operators in Hilbert space. Vol2, *Pitman publishing Inc. Boston, London, Melbourne*

[15] Newton R.G. (1966), Scattering theory of waves and particles, *Mcgraw-Hill, Inc. New York San Francisco*

[16] Bolle D. and Gesztesy F. (1984), Scattering observables in arbitrary dimension n≥2, *Phys.Rev.A*30, 1279-1293

[17] Gesztesy F. and Thaller B. (1981), Born expressions for Coulomb type interactions, *J. Phys. A*14, 639-657

[18] Abramowitz M. and Stegurn I.A. (1972), Handbook of Mathematical functions, *Dover publication Inc, New-York N.Y*

[19] Vyabandi A. and Shabani J. (2020), Scattering Theory for 2N parameter models of finitely many relativistic δ -sphere and relativistic δ –sphere plus coulomb interactions supported by concentric spheres in quantum mechanics, *Advances in Math*. Phys. 2020