Volume **28** (2020) 36-44 http://revue.ub.edu.bi/JUB

Two parameters nonrelativistic δ' -sphere interaction in quantum mechanics

Alfred Vyabandi

Université du Burundi, Institut de Pédagogie Appliquée (IPA), B.P 5223, Bujumbura - BURUNDI

E-mail: alfred.vyabandi@ub.edu.bi

Received: January 2013
Accepted for publication: on August 30th, 2013
Published online for the first time: on March 16th, 2020

Abstract

We introduce and make a systematic study of two new exactly solvable models of two parameters nonrelativistic δ'-sphere and δ' -sphere plus Coulomb interactions which constitute a generalization of one parameter δ' interactions formally given by $H = -\Delta + \frac{\gamma_n}{|x|} + \alpha\, \delta'(|x| - R), x \in \mathbb{R}^3, R > 0, n = 1,2, \gamma_1 = 0,\ \gamma_2 \neq 0$[1] . We provide the mathematical definitions of hamiltonians and obtain new results for both model, in particular the resolvents equations, spectral properties and some scattering quantities theory.

Keywords: Equation résolvante, propriétés spectrales, théorie de diffusion.

Résumé

Nous étudions deux nouveaux modèles à deux paramètres d'interactions δ'-sphérique et δ'-sphérique plus une interaction Coulombienne qui constituent une généralisation des interactions de surface à un paramètre formellement données par $H = -\Delta + \frac{\gamma_n}{|x|} + \alpha\, \delta'(|x| - R), x \in \mathbb{R}^3, R > 0, n = 1,2, \gamma_1 = 0,\ \gamma_2 \neq 0$[1]. Nous définissons les hamiltoniens de ces modèles, calculons les équations résolvantes, étudions les propriétés spectrales et déterminons les éléments de la théorie de diffusion.

Keywords: resolvent equation, spectral property, diffusion theory.

1. Introduction

The study of sphere interactions is interesting both from the mathematical point of view and for understanding various physical phenomena in quantum mechanics [1]-[13]. To the best of our knowledge, the δ'-sphere interaction have a recent history [1]. They are exactly solvable models and their study aim at extending a well understanding of contacts interactions. The δ'-sphere interaction was introduced by J. Shabani et Al [1] and is different from the δ-sphere interaction of the second type introduced in ref [2] and inadequately called "δ' $-$sphere interaction".
In this paper, we discuss the basic properties of new exactly solvable models of two parameters nonrelativistic pure δ' $-$sphere δ' $-$sphere interaction plus Coulomb interaction in tree space dimensions. The two parameters nonrelativistic δ' $-$sphere interaction generalizes the nonrelativistic δ' $-$sphere interaction of the first and the second type. [1]

The paper is organized as follow: In Sec.II, we provide a mathematical definition of the Hamiltonian, the resolvent equation, the spectral properties and the scattering data (the phase shift, scattering matrix, scattering amplitude and scattering cross section). In Sec III, we generalize the results of Sec.II to the case of a two parameters δ' $-$sphere interaction plus a Coulomb interaction.

2. Basic properties of the nonrelativistic two parameters δ' −sphere interaction

2.1 Definition of the model

In this section, using the theory of self adjoint extensions of symmetric closed operators, we provide the mathematical definition of quantum hamiltonian describing the nonrelativistic two parameters δ' −sphere interaction.
Consider the formal expression

$$H = -\Delta + \alpha\delta'(|x| - R), \quad x \in \mathbb{R}^3, R > 0. \quad (2.1)$$

We define in $L^2(\mathbb{R}^3)$ the closed symmetric and non negative operator by

$$\dot{H} = -\Delta,$$

$$\mathcal{D}(\dot{H}) = \{f \in H^{2,2}(\mathbb{R}^3) / f(\partial\overline{k(0,R)}) = f'(\partial\overline{k(0,R)}) = 0\} \quad (2.2)$$

where $H^{m,n}(\Omega)$ is the Sobolev space of indices (m, n) and $\overline{k(0,R)}$ is the closed ball of radius R centered at the origin in $\mathbb{R}^3$.
We decompose the state Hilbert space $L^2(\mathbb{R}^3)$ with respect to angular momenta by:

$$L^2(\mathbb{R}^3) = L^2((0,\infty), r^2 dr) \otimes L^2(S^2). \quad (2.3)$$

We introduce the unitary transformation by

$$U : L^2((0,\infty), r^2 dr) \to L^2((0,\infty), dr) \equiv L^2((0;\infty)),$$
$$f \to (Uf)(r) = rf(r) \quad (2.4)$$

which enables us to represent $L^2(\mathbb{R}^3)$ by

$$L^2(\mathbb{R}^3) = \oplus_{l=0}^{\infty} U^{-1} L^2((0,\infty)) \otimes [Y_l^{-l} \dots Y_l^{-l}] \quad (2.5)$$

where the spherical harmonics $Y_l^m, l \in \mathbb{N}_0, -l \le m \le l$, provide a basis for $L^2(S^2)$ (S^2 is the unit sphere in $\mathbb{R}^3$). [...] denotes the linear span of vectors in $L^2(S^2)$.
With respect to the decomposition (2.5), $\dot{H}$ reads:

$$\dot{H} = \oplus_{l=0}^{\infty} U^{-1} \dot{h}_l U \otimes 1\mathrm{I} \quad (2.6)$$

where

$$\dot{h}_l = -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2}$$
$$\mathcal{D}(\dot{h}_l) = \{f \in L^2((0,\infty)) / f, f' \in AC_{loc}((0,\infty)); f(0+) = 0 \text{ if } l = 0; f(R\pm) = f'(R\pm) = 0, -f'' + l(l+1)r^{-2}f \in L^2((0,\infty))\}. \quad (2.7)$$

$AC_{loc}((\Omega))$ denotes the set of locally absolutely continuous functions on $\Omega \subset \mathbb{R}$ and $f(x\pm) = \lim_{\epsilon\to 0} f(x\pm\epsilon)$. The adjoint $\dot{H}^*$ of $\dot{H}$ is given by

$$\dot{H}^* = \oplus_{l=0}^{\infty} U^{-1} \dot{h}_l^* U \otimes 1\mathrm{I} \quad (2.8)$$

Where the adjoint $\dot{h}_l^*$ of $\dot{h}_l$ reads :

$$\dot{h}_l^* = -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2}$$
$$\mathcal{D}(\dot{h}_l^*) = \{f \in L^2((0,\infty)) / f, f' \in AC_{loc}((0,\infty) \setminus \{R\}); -f'' + l(l+1)r^{-2}f \in L^2((0,\infty))\}. \quad (2.9)$$

A straightforward computation shows that the equation

$$\dot{h}_l^* \psi_l(k) = k^2 \psi_l(k), \psi_l \in \mathcal{D}(\dot{h}_l^*), \operatorname{Im} k > 0, l \in \mathbb{N}_0 \quad (2.10)$$

has two linearly independent solutions :

$$\phi_l^{(1)}(k,r) = \begin{cases} F_l(k,r); & r < R \\ 0; & r > R \end{cases} \quad (2.11)$$

$$\phi_l^{(2)}(k,r) = \begin{cases} 0; & r < R \\ G_l(k,r); & r > R \end{cases} \quad (2.12)$$

where

$$F_l(k,r) = \Gamma\left(l + \frac{3}{2}\right)\left(\frac{k}{2}\right)^{-l-\frac{1}{2}} r^{\frac{1}{2}} J_{l+\frac{1}{2}}(kr)$$

$$G_l(k,r) = \mathrm{i}\frac{\pi}{2}\Gamma\left(l + \frac{3}{2}\right)\left(\frac{k}{2}\right)^{-l-\frac{1}{2}} r^{\frac{1}{2}} H_{l+\frac{1}{2}}^{(1)}(kr) \quad (2.13)$$

and $J_\nu(.)$ denote the Bessel function and $H_\nu^{(1)}(.)$ the Hankel function of the first type of order ν. Therefore, $\dot{h}_l$ has defiency indices (2,2) and consequently, all self-adjoint(s.a) extensions of $\dot{h}_l$ are given by a 4- parameters family of self adjoint operators [14].
Let us define a special 2-parameters boundary conditions at r = R.
Consider the radial Schrödinger equation for a δ'-sphere interaction given by the formal expression

$$\left(-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + \alpha_l\delta'(r-R)\right)\phi = k^2\phi. \quad (2.14)$$

Suppose that the derivate ϕ' is discontinuous at r = R. Integrating equation (2.14) over $[R-\varepsilon, R+\varepsilon]$ and taking the limit $\varepsilon \to 0+$, we obtain

$$\left(1 + \frac{\alpha_l}{2}\right)\phi'(R+) + \left(\frac{\alpha_l}{2} - 1\right)\phi'(R-) = 0. \quad (2.15)$$

We have used the relation [1]:

$$\lim_{\varepsilon\to 0}\int_{R-\varepsilon}^{R+\varepsilon} \delta'(r-R)\phi(r)dr = -\frac{1}{2}(\phi'(R+) + \phi'(R-)). \quad (2.16)$$

Interchanging ϕ and ϕ' in equation eq. (2.15) and taking $\alpha_l \longrightarrow \beta_l$, we obtain the second boundary condition:

$$\left(1 + \frac{\beta_l}{2}\right)\phi(R+) + \left(\frac{\beta_l}{2} - 1\right)\phi(R-) = 0. \quad (2.17)$$

Therefore the boundary conditions which define the two parameters nonrelativistic δ'-sphere interaction are:

$$\begin{cases} \left(1 + \frac{\alpha_l}{2}\right)\phi'(R+) + \left(\frac{\alpha_l}{2} - 1\right)\phi'(R-) = 0 \\ \left(1 + \frac{\beta_l}{2}\right)\phi(R+) + \left(\frac{\beta_l}{2} - 1\right)\phi(R-) = 0. \end{cases}$$

We consider a special two parameters family h_{l,α_l,β_l} of (s.a) extensions of $\dot{h}_l$ defined by

$$h_{l,\alpha_l,\beta_l} = -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2},$$

$$\mathcal{D}(h_{l,\alpha_l,\beta_l}) = \left\{\phi \in \mathcal{D}(\dot{h}_l^*) \left| \begin{matrix} \left(1 + \frac{\alpha_l}{2}\right)\phi'(R+) + \left(\frac{\alpha_l}{2} - 1\right)\phi'(R-) = 0 \\ \left(1 + \frac{\beta_l}{2}\right)\phi(R+) + \left(\frac{\beta_l}{2} - 1\right)\phi(R-) = 0 \end{matrix} \right.\right\},$$

$\alpha_l, \beta_l \in \mathbb{R}$. (2.19)

The case $\alpha_l = \beta_l = 0$ in eq. (2.19) coincides with the free kinetic energy hamiltonian $h_{l,0}$ for a fixed angular momentum l. Let $\alpha=\{\alpha_l\}_{l\in\mathbb{N}_0}, \beta=\{\beta_l\}_{l\in\mathbb{N}_0}$ and introduce in $L^2(\mathbb{R}^3)$ the operator:

$$H_{\alpha,\beta}=\oplus_{l=0}^{\infty} U^{-1} h_{l,\alpha_l,\beta_l} U \otimes 1\mathrm{I}. \tag{2.20}$$

The hamiltonian $H_{\alpha,\beta}$ provides a mathematical definition of the two parameters nonrelativistic δ'-sphere interaction.
The particular case $\alpha=\beta=0$ in eq. (2.20) leads to the free hamiltonian

$$H_0=-\Delta; \mathcal{D}(H_0)=H^{2,2}(\mathbb{R}^3). \tag{2.21}$$

The particular cases $\alpha\neq 0, \beta=0$ and $\alpha=0, \beta\neq 0$ in eq. (2.20) yield the nonrelativistic δ'-sphere interactions of the first and second type respectively[1].

2.2 The resolvent equation

Theorem 2.1: The resolvent of $h_{l,\alpha_l,\beta_l}$and $H_{\alpha,\beta}$read respectively:

(i)
$$\begin{aligned}\left(h_{l,\alpha_l,\beta_l}-k^2\right)^{-1} &= \left(h_{l,0}-k^2\right)^{-1} + \mu_l(k)\Big[\alpha_l(\tilde{\psi}_l^{(2)}(-\bar{k}),.)\psi_l^{(2)}(k) - \\ &\beta_l(\psi_l^{(2)}(-\bar{k}),.)\tilde{\psi}_l^{(2)}(k) - \frac{\alpha_l\beta_l}{2}(\psi_l^{(2)}(-\bar{k}),.)\tilde{\psi}_l^{(1)}(k) + \frac{\alpha_l\beta_l}{2}(\tilde{\psi}_l^{(2)}(-\bar{k}),.)\psi_l^{(1)}(k)\Big],\\ & k^2\in\rho(h_{l,\alpha_l,\beta_l});\ \mathrm{Im}\,k>0;\ l\in\mathbb{N}_0\end{aligned} \tag{2.22}$$

where

$$\mu_l(k)=\left\{1-\frac{\alpha_l\beta_l}{4}+\frac{1}{2}(\beta_l-\alpha_l)[G_l(k,R)F_l(k,R)]'\right\}^{-1} \tag{2.23}$$

$$\psi_l^{(n)}(k,r)=\begin{cases} G_l(k,R)F_l(k,r); & r<R\\ (-1)^n F_l(k,R)G_l(k,r), & r>R, n=1,2\end{cases} \tag{2.24}$$

$$\tilde{\psi}_l^{(n)}(k,r)=\begin{cases} G_l'(k,R)F_l(k,r), & r<R\\ (-1)^n F_l'(k,R)G_l(k,r), & r>R, n=1,2\end{cases} \tag{2.25}$$

(ii)
$$\begin{aligned}\left(H_{\alpha,\beta}-k^2\right)^{-1} &= (H_0-k^2)^{-1} + \oplus_{l=0}^{\infty}\oplus_{m=-l}^{l}\mu_l(k)\Big[\alpha_l\big(|.|\tilde{\psi}_l^{(2)}(-\bar{k})Y_l^{(m)},.\big)\mathrm{x}\\ &|.|\psi_l^{(2)}(k)Y_l^{(m)} - \beta_l\big(|.|\psi_l^{(2)}(-\bar{k})Y_l^{(m)},.\big)|.|\tilde{\psi}_l^{(2)}(k)Y_l^{(m)} - \frac{\alpha_l\beta_l}{2}\big(|.|\psi_l^{(2)}(-\bar{k})Y_l^{(m)},.\big)\mathrm{x}\\ &|.|\tilde{\psi}_l^{(1)}(k)Y_l^{(m)} + \frac{\alpha_l\beta_l}{2}\big(|.|\tilde{\psi}_l^{(2)}(-\bar{k})Y_l^{(m)},.\big)|.|\psi_l^{(1)}(k)Y_l^{(m)}\Big], k^2\\ &\in\rho(H_{\alpha,\beta});\ \mathrm{Im}\,k>0,\quad l\in\mathbb{N}_0\end{aligned} \tag{2.26}$$

Proof:
Since the symmetric operator $\dot{h}_l$ has deficiency indices (2,2), the Krein's resolvent formula of h_{l,α_l,β_l} reads

$$\begin{aligned}\left(h_{l,\alpha_l,\beta_l}-k^2\right)^{-1} &= \left(h_{l,0}-k^2\right)^{-1} + \sum_{i,n=0}^{2}\lambda_{in}(k)\left(\phi_l^{(n)}(-\bar{k}),.\right)\phi_l^{(i)}(k), k^2\\ &\in\rho(h_{l,\alpha_l,\beta_l}),\quad \mathrm{Im}\,k>0, l\in\mathbb{N}_0\end{aligned} \tag{2.27}$$

where $\phi_l^{(n)}$, n = 1; 2 are given by eq. (2.11) and eq. (2.12). Let $g\in L^2((0;\infty))$ and define the function $f(k,r)$ by

$$f(k,r)=\left(\left(h_{l,\alpha_l,\beta_l}-k^2\right)^{-1}g\right)(r). \tag{2.28}$$

Since $f\in\mathcal{D}(h_{l,\alpha_l,\beta_l})$, it follows that f satisfies the boundary conditions in eq. (2.19).The implementation of these boundary conditions gives the constants λ_{in}:

$$\begin{aligned}\lambda_{11}(k)&=\mu_l(k)[\alpha_l G_l'(k,R)G_l(k,R) - \beta_l G_l'(k,R)G_l(k,R)]\\ \lambda_{12}(k)&=\mu_l(k)\left[-\alpha_l F_l'(k,R)G_l(k,R)+\beta_l G_l'(k,R)F_l(k,R) - \frac{\alpha_l\beta_l}{2}\right]\\ \lambda_{22}(k)&=\mu_l(k)[-\alpha_l F_l'(k,R)F_l(k,R) + \beta_l F_l'(k,R)F_l(k,R)]\\ \lambda_{21}(k)&=\mu_l(k)\left[\alpha_l F_l'(k,R)G_l(k,R) - \beta_l G_l'(k,R)F_l(k,R)+\frac{\alpha_l\beta_l}{2}\right]\end{aligned} \tag{2.29}$$

Inserting them into eq. (2.27) we obtain eq. (2.22).
Eq. (2.26) follows from the decomposition eq. (2.20) and eq. (2.22).
Let us now provide the additional information on the domain of $h_{l,\alpha_l,\beta_l}$and prove that the two parameters δ' interaction is local.

Theorem 2.2: The domain $\mathcal{D}(h_{l,\alpha_l,\beta_l}), -\infty<\alpha_l,\beta_l\leq\infty, r\in\mathbb{R}$, consists of functions of the type :

$$\begin{aligned}\phi_{l,\alpha_l,\beta_l}(k,r) &= \chi_l(k,r) + \mu_l(k)\Big(\alpha_l\chi_l'(k,R)\psi_l^{(2)}(k,r) - \beta_l\chi_l(k,R)\tilde{\psi}_l^{(2)}(k,r) - \\ &\frac{\alpha_l\beta_l}{2}\chi_l(k,R)\tilde{\psi}_l^{(1)}(k,r) + \frac{\alpha_l\beta_l}{2}\chi_l'(k,R)\psi_l^{(1)}(k,r)\Big)\end{aligned} \tag{2.30}$$

where $\mu_l(k)$ is defined by eq. (2.23), $\chi_l\in\mathcal{D}(h_{l,0})$, and $k^2\in\rho(h_{l,\alpha_l,\beta_l})$, Imk > 0. The decomposition (2.30) is unique and with $\phi_{l,\alpha_l,\beta_l}$ of this form, we obtain:

$$(h_{l,\alpha_l,\beta_l}-k^2)\phi_{l,\alpha_l,\beta_l}=(h_{l,0}-k^2)\chi_l. \tag{2.31}$$

Consider $\phi_{l,\alpha_l,\beta_l}\in\mathcal{D}(h_{l,\alpha_l,\beta_l})$ and $\phi_{l,\alpha_l,\beta_l}=0$ in a open set $\Lambda\subset[0,\infty)$, then $h_{l,\alpha_l,\beta_l}\phi_{l,\alpha_l,\beta_l}=0$ in Λ, which means that the interaction described by h_{l,α_l,β_l} is local.

Proof:
The domain $\mathcal{D}(h_{l,\alpha_l,\beta_l})$ reads:

$$\mathcal{D}(h_{l,\alpha_l,\beta_l})=\left(h_{l,\alpha_l,\beta_l}-k^2\right)^{-1}(h_{l,0}-k^2)\mathcal{D}(h_{l,0})$$

$$= \Big\{ \big(h_{l,0} - k^2\big)^{-1} + \mu_l(k)\big[\alpha_l\big(\tilde{\psi}_l^{(2)}(-\bar{k}),.\big)\psi_l^{(2)}(k) - \beta_l\big(\psi_l^{(2)}(-\bar{k}),.\big)\tilde{\psi}_l^{(2)}(k) - \frac{\alpha_l\beta_l}{2}\big(\psi_l^{(2)}(-\bar{k}),.\big)\tilde{\psi}_l^{(1)}(k) + \frac{\alpha_l\beta_l}{2}\big(\tilde{\psi}_l^{(2)}(-\bar{k}),.\big)\psi_l^{(1)}(k)\big]\Big\}\big(h_{l,0} - k^2\big)\mathcal{D}\big(h_{l,0}\big),$$

$k^2 \in \rho\big(h_{l,\alpha_l,\beta_l}\big)$; Im $k > 0, l \in \mathbb{N}_0$. (2.32)

Eq. (2.32) prove eq. (2.30). Let now $\phi_{l,\alpha_l,\beta_l} = 0$ in eq. (2.30), then

$$\chi_l(k,r) = \mu_l(k)\Big(-\alpha_l\chi_l'(k,R)\psi_l^{(2)}(k,r) + \beta_l\chi_l(k,R)\tilde{\psi}_l^{(2)}(k,r) + \frac{\alpha_l\beta_l}{2}\chi_l(k,R)\tilde{\psi}_l^{(1)}(k,r) - \frac{\alpha_l\beta_l}{2}\chi_l'(k,R)\psi_l^{(1)}(k,r)\Big), \quad (2.33)$$

and $\chi_l \in AC_{loc}((0,\infty))$ implies $\chi_l = 0$ which prove the uniquess of eq. (2.30). The relation

$$\big(h_{l,\alpha_l,\beta_l} - k^2\big)^{-1}\big(h_{l,0} - k^2\big)\chi_l = \chi_l + \mu_l(k)\big[\alpha_l\big(\tilde{\psi}_l^{(2)}(-\bar{k}),\big(h_{l,0} - k^2\big)\chi_l\big)\psi_l^{(2)}(k) - \beta_l\big(\psi_l^{(2)}(-\bar{k}),\big(h_{l,0} - k^2\big)\chi_l\big)\tilde{\psi}_l^{(2)}(k) - \frac{\alpha_l\beta_l}{2}\big(\psi_l^{(2)}(-\bar{k}),\big(h_{l,0} - k^2\big)\chi_l\big)\tilde{\psi}_l^{(1)}(k) + \frac{\alpha_l\beta_l}{2}\big(\tilde{\psi}_l^{(2)}(-\bar{k}),\big(h_{l,0} - k2\chi l\psi l1k$$

$= \phi_{l,\alpha_l,\beta_l}, k^2 \in \rho\big(h_{l,\alpha_l,\beta_l}\big)$, Im $k > 0$ (2.34)

prove (2.31).

Next prove now the interaction h_{l,α_l,β_l} is local.

(a) We assume $R \notin \Lambda$ and

$$\big(h_{l,0} - k^2\big)\Big(-\alpha_l\chi_l'(k,R)\psi_l^{(2)}(k,.) + \beta_l\chi_l(k,R)\tilde{\psi}_l^{(2)}(k,.) + \frac{\alpha_l\beta_l}{2}\chi_l(k,R)\tilde{\psi}_l^{(1)}(k,.) - \frac{\alpha_l\beta_l}{2}\chi_l'(k,R)\psi_l^{(1)}(k,.)\Big)(r) = 0 \quad (2.35)$$

From eq. (31) we can write

$$\big(h_{l,\alpha_l,\beta_l}\phi_{l,\alpha_l,\beta_l}\big)(r) = k^2\phi_{l,\alpha_l,\beta_l}(r) + \Big(\big(h_{l,0} - k^2\big)\chi_l\Big)(r) \quad (2.36)$$

If we assume that $\phi_{l,\alpha_l,\beta_l} = 0$ in eq. (2.36) we obtain

$$\big(h_{l,\alpha_l,\beta_l}\phi_{l,\alpha_l,\beta_l}\big)(r) = \mu_l(k)\Big\{\big(h_{l,0} - k^2\big)\big(-\alpha_l\chi_l'(k,R)\psi_l^{(2)}(k,.) + \beta_l\chi_l(k,R)\tilde{\psi}_l^{(2)}(k,.) + \frac{\alpha_l\beta_l}{2}\chi_l(k,R)\tilde{\psi}_l^{(1)}(k,.) - \frac{\alpha_l\beta_l}{2}\chi_l'(k,R)\psi_l^{(1)}(k,.)\big)\Big\}(r) = 0, r \in \Lambda. \quad (2.37)$$

(b) If $R \in \Lambda$ then $\phi_{l,\alpha_l,\beta_l}(R) = 0$ and $\chi_l \in AC_{loc}((0;\infty))$ implies $\chi_l = 0$ and hence

$$\big(h_{l,\alpha_l,\beta_l}\phi_{l,\alpha_l,\beta_l}\big)(r) = k^2\phi_{l,\alpha_l,\beta_l}(r) = 0, r \in \Lambda. \quad (2.38)$$

We proved that if $\phi_{l,\alpha_l,\beta_l} = 0$ in a open set $\Lambda \subset \mathbb{R}$, then $h_{l,\alpha_l,\beta_l}\phi_{l,\alpha_l,\beta_l} = 0$ in Λ, which mean that the interaction h_{l,α_l,β_l} is local.

2.3 Spectral properties

The spectral properties of h_{l,α_l,β_l} are provided by the following theorem where σ_{ess}; σ_{ac}; σ_{sc} denote the essential spectrum, absolutely continuous spectrum, singularly continuous spectrum respectively.

Theorem 2.3: For all $\alpha_l, \beta_l \in (0,\infty)$ we have:

$$\sigma_{ess}\big(h_{l,\alpha_l,\beta_l}\big) = \sigma_{ac}\big(h_{l,\alpha_l,\beta_l}\big) = [0,\infty), \quad (2.39)$$

$$\sigma_{sc}\big(h_{l,\alpha_l,\beta_l}\big) = \emptyset. \quad (2.40)$$

Negative eigenvalues of h_{l,α_l,β_l} are obtained from the equation

$$1 - \frac{\alpha_l\beta_l}{4} + \frac{1}{2}(\beta_l - \alpha_l)\psi_l^{(2)'}\big(\mathrm{i}\sqrt{-E},R\big) = 0, E < 0. \quad (2.41)$$

Proof :

Eq. (2.39) and eq. (2.40) follow from Weyl's theorem [16], p.112 and theorem XIII respectively.

Eq. (2.41) follows from eq. (2.22).

The resonances of h_{l,α_l,β_l} are defined as poles of the resolvent eq. (2.22) in the unphysical sheet

Im k < 0.

2.4 Scattering theory for the pair (h_{l,α_l,β_l} , $h_{l,0}$)

For $k \geq 0$, we define the function

$$\phi_{l,\alpha_l,\beta_l}(k,r) = \mathrm{F_l}(\mathrm{k,r}) + \mu_l(k)\Big(\alpha_l F_l'(k,R)\psi_l^{(2)}(k,r) - \beta_l F_l(k,R)\tilde{\psi}_l^{(2)}(k,r) - \frac{\alpha_l\beta_l}{2}F_l(k,R)\tilde{\psi}_l^{(1)}(k,r) + \frac{\alpha_l\beta_l}{2}F_l'(k,R)\psi_l^{(1)}(k,r)\Big). \quad (2.42)$$

One can show easly that the function $\phi_{l,\alpha_l,\beta_l}$ fulfills the following conditions:

$$\Big(1 + \frac{\alpha_l}{2}\Big)\phi_{l,\alpha_l,\beta_l}'(R+) + \Big(\frac{\alpha_l}{2} - 1\Big)\phi_{l,\alpha_l,\beta_l}'(R-) = 0$$

$$\Big(1 + \frac{\beta_l}{2}\Big)\phi_{l,\alpha_l,\beta_l}(R+) + \Big(\frac{\beta_l}{2} - 1\Big)\phi_{l,\alpha_l,\beta_l}(R-) = 0,$$

$$\phi_{l,\alpha_l,\beta_l}''(k,r) + l(l+1)r^{-2}\phi_{l,\alpha_l,\beta_l}(k,r) = k^2\phi_{l,\alpha_l,\beta_l}(k,r) \quad (2.43)$$

Therefore $\phi_{l,\alpha_l,\beta_l}$ are the scattering wave functions of h_{l,α_l,β_l}. The phase shift of h_{l,α_l,β_l} is obtained through the asymptotic behavior of h_{l,α_l,β_l} as $r \longrightarrow \infty$.

For $k > 0$, one has [17]

$$\phi_{l,\alpha_l,\beta_l}\overrightarrow{r \longrightarrow \infty}\, A_l(k)\sin\left(kr - \frac{l\pi}{2}\right)$$
$$+ \mu_l(k)\left\{\alpha_l F_l^{'}(k,R)F_l(k,R)B_l(k)\exp\left[-\mathrm{i}\left(kr - \frac{l\pi}{2}\right)\right] - \beta_l F_l^{'}(k,R)F_l(k,R)B_l(k)\exp\left[-\mathrm{i}\left(kr - \frac{l\pi}{2}\right)\right]\right\}$$
$$= [A_l(k) - \mathrm{i}\mu_l(k)B_l(k)(\alpha_l - \beta_l)\alpha_l F_l^{'}(k,R)F_l(k,R)]\sin\left(kr - \frac{l\pi}{2}\right) - \mu_l(k)B_l(k)(\alpha_l - \beta_l)\alpha_l F_l^{'}(k,R)F_l(k,R)\cos\left(kr - \frac{l\pi}{2}\right)$$
$$= [L_{1,l}^2(k) + L_{2,l}^2(k)]^{\frac{1}{2}}\sin\left(kr - \frac{l\pi}{2} + \delta_{l,\alpha_l,\beta_l}(k)\right) + 0(1). \qquad (2.44)$$

Therefore the phase shifts reads:

$$\delta_{l,\alpha_l,\beta_l}(k) = -\arctan\frac{L_{2,l}}{L_{1,l}}$$
$$= -\arctan\frac{\mu_l(k)B_l(k)(\alpha_l - \beta_l)F_l^{'}(k,R)F_l(k,R)}{A_l(k) - \mathrm{i}\mu_l(k)B_l(k)(\alpha_l - \beta_l)F_l^{'}(k,R)F_l(k,R)} \qquad (2.45)$$

where [17]

$$A_l(k) = 2^{-l}k^{-l-1}\Gamma(+2)\Gamma(+1)^{-1}, \qquad (2.46)$$
$$B_l(k) = \frac{1}{k\,A_l(k)}. \qquad (2.47)$$

The on-shell scattering matrix is given by

$$S_{l,\alpha_l,\beta_l}(k) = \exp(2\mathrm{i}\delta_{l,\alpha_l,\beta_l}(k))$$
$$= 1 - 2\mathrm{i}kB_l^2(k)\mu_l(k)(\alpha_l - \beta_l)F_l^{'}(k,R)F_l(k,R). \qquad (2.48)$$

The partial wave scattering amplitude is given by:

$$f_{l,\alpha_l,\beta_l}(k) = \frac{\exp\left(2\mathrm{i}\delta_{l,\alpha_l,\beta_l}(k)\right) - 1}{2\mathrm{i}k} = -B_l^2(k)\mu_l(k)(\alpha_l - \beta_l)F_l^{'}(k,R)F_l(k,R). \qquad (2.49)$$

The on-shell scattering amplitude $f_{l,\alpha_l,\beta_l}(k,\omega,\omega^{'})$ associated with $H_{\alpha,\beta}$ reads:

$$f_{l,\alpha_l,\beta_l}(k,\omega,\omega^{'}) = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{\exp\left(2\mathrm{i}\delta_{l,\alpha_l,\beta_l}(k)\right) - 1}{2\mathrm{i}k}\overline{Y_l^m(\omega^{'})}Y_l^m(\omega). \qquad (2.50)$$

The corresponding effective range expansion reads:

$$[(2l + 1)!!]^2k^{2l+1}\cot\delta_{l,\alpha_l,\beta_l}(k) = -\,a_{l,\alpha_l,\beta_l}^{-1} + \frac{1}{2}r_{l,\alpha_l,\beta_l}k^2 + 0(k^4) \qquad (2.51)$$

where the scattering length a_{l,α_l,β_l} is given by:

$$a_{l,\alpha_l,\beta_l} = \mu_l(0)(\alpha_l - \beta_l)(l+1)R^{2l+1}. \qquad (2.52)$$

The total scattering section for the pair (h_{l,α_l,β_l} ,$h_{l,0}$) is given by

$$\sigma_{total} = \int d\Omega\sigma(\theta,\varphi) \qquad (2.53)$$

where θ,φ are angulaire variables. The differential scattering cross section σ reads[18]

$$\sigma = 4\sum_{l=0}^{\infty}\left(l + \frac{1}{2}\right)^2\left|f_{l,\alpha_l,\beta_l}\right|^2 P_l^2(\cos\theta). \qquad (2.54)$$

The Legendre Polynomials $P_l(\cos\theta)$ are defined by Rodrigues formula [19]:

$$P_l(\cos\theta) = \frac{(-)^l}{2^l l!}\left(\frac{\mathrm{d}}{\mathrm{d}\cos\theta}\right)(\sin\theta)^{2l}. \qquad (2.55)$$

Then eq. (53) reads:

$$\sigma_{total} = 4\sum_{l=0}^{\infty}\left(l + \frac{1}{2}\right)\left|f_{l,\alpha_l,\beta_l}\right|^2\int \mathrm{d}\Omega\left(l + \frac{1}{2}\right)P_l(\cos\theta)\,P_l(\cos\theta)$$
$$= 4\sum_{l=0}^{\infty}\left(l + \frac{1}{2}\right)\left|f_{l,\alpha_l,\beta_l}\right|^2\int_0^{2\pi}d\varphi\int_0^{\pi}\mathrm{d}\theta\sin\theta\left(l + \frac{1}{2}\right)P_l(\cos\theta)P_l(\cos\theta)$$
$$= -8\pi\sum_{l=0}^{\infty}\left(l + \frac{1}{2}\right)\left|f_{l,\alpha_l,\beta_l}\right|^2\int_0^{\pi}\mathrm{d}(\cos\theta)\left(l + \frac{1}{2}\right)P_l(\cos\theta)P_l(\cos\theta)$$
$$= 8\pi\sum_{l=0}^{\infty}\left(l + \frac{1}{2}\right)\left|f_{l,\alpha_l,\beta_l}\right|^2\int_{-1}^{1}\mathrm{d}x\left(l + \frac{1}{2}\right)P_l(x)P_l(x) \qquad (2.56)$$

with x = $\cos\theta$, then, the total scattering cross section reads:

$$\sigma_{total} = 8\pi\sum_{l=0}^{\infty}\left(l + \frac{1}{2}\right)\left|f_{l,\alpha_l,\beta_l}\right|^2$$
$$= 8\pi\sum_{l=0}^{\infty}\left(l + \frac{1}{2}\right)\left|\frac{(\alpha_l - \beta_l)B_l^2(k)F_l^{'}(k,R)F_l(k,R)}{1 - \frac{\alpha_l\beta_l}{4} + \frac{1}{2}(\beta_l - \alpha_l)[G_l(k,R)F_l(k,R)]^{'}}\right|^2. \qquad (2.57)$$

A straightforward computation shows that:

$$\sigma_{total} = \sum_{l=0}^{\infty}\sigma_l \qquad (2.58)$$

where σ_l is called partial cross sections and is given by :

$$\sigma_l = 4\pi(2l+1)k^{-2}\sin^2\delta_{l,\alpha_l,\beta_l}.$$

3. Basic properties of the two-parameters nonrelativistic δ′-sphere interaction plus Coulomb interaction

3.1. Definition of the model

The two-parameters $\boldsymbol{\delta}^{'}$-sphere interaction plus Coulomb interaction is formally given by the hamiltonian

$$H = -\Delta + \frac{\gamma}{|x|} + \alpha\delta'(|x| - R), \ \gamma \in \mathbb{R}, \ x \in \mathbb{R}^3, R > 0. \quad (3.1)$$

Consider in $L^2(\mathbb{R}^3)$ the operator

$$\dot{H}_\gamma = -\Delta + \frac{\gamma}{|x|},$$

$$\mathcal{D}(\dot{H}_\gamma) = \{f \epsilon H^{2,2}(\mathbb{R}^3)/f(\partial \overline{k(0,R)}) = f'(\partial \overline{k(0,R)}) = 0\}. \quad (3.2)$$

We introduce the operator $\dot{H}_\gamma$ by

$$\dot{H}_\gamma = \oplus_{l=0}^{\infty} U^{-1} \dot{h}_{l,\gamma} U \otimes 1\mathrm{I} \quad (3.3)$$

where

$$\dot{h}_{l,\gamma} = -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + \frac{\gamma}{r}$$

$$\mathcal{D}(\dot{h}_{l,\gamma}) = \{f \in L^2((0,\infty))/f, f' \in AC_{loc}((0,\infty));$$
$$f(0+) = 0 \text{ if } l = 0; f(R\pm) = f'(R\pm) = 0,$$
$$-f'' + l(l+1)r^{-2}f + \gamma r^{-1} f \in L^2((0,\infty))\}, \gamma \in \mathbb{R}, l \in \mathbb{N}_0. \quad (3.4)$$

The adjoint $\dot{H}_\gamma^*$ of $\dot{H}_\gamma$ reads

$$\dot{H}_\gamma^* = \oplus_{l=0}^{\infty} U^{-1} \dot{h}_{l,\gamma}^* U \otimes 1\mathrm{I} \quad (3.5)$$

where the adjoint $\dot{h}_{l,\gamma}^*$ of $\dot{h}_{l,\gamma}$ reads:

$$\dot{h}_{l,\gamma}^* = -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + \frac{\gamma}{r}$$

$$\mathcal{D}(\dot{h}_{l,\gamma}^*) = \{f \in L^2((0,\infty))/f, f' \in AC_{loc}((0,\infty) \setminus \{R\});$$
$$f(0+) = 0 \text{ if } l = 0;$$
$$-f'' + l(l+1)r^{-2}f + \gamma r^{-1} f \in L^2((0,\infty))\}, \gamma \in \mathbb{R}, l \in \mathbb{N}_0. \quad (3.6)$$

The deficiency indices equation

$$\dot{h}_{l,\gamma}^* \phi_{l,\gamma}(k) = k^2 \phi_{l,\gamma}(k), \phi_{l,\gamma} \in \mathcal{D}(\dot{h}_{l,\gamma}^*), \mathrm{Im}\, k > 0, l \in \mathbb{N}_0 \quad (3.7)$$

has two linearly independent solutions :

$$\phi_{l,\gamma}^{(1)}(k,r) = \begin{cases} F_{l,\gamma}(k,r); & r < R \\ 0 & r > R \end{cases} \quad (3.8)$$

$$\phi_{l,\gamma}^{(2)}(k,r) = \begin{cases} 0; & r < R \\ G_{l,\gamma}(k,r) & r > R \end{cases} \quad (3.9)$$

where

$$F_{l,\gamma}(k,r) = r^{l+1} \exp(ikr)\, {}_1F_1\left(l + 1 + \frac{i\gamma}{2k}; 2l + 2; -2ikr\right)$$

$$G_{l,\gamma}(k,r) = \Gamma(2l+2)^{-1} \Gamma\left(l + 1 + \frac{i\gamma}{2k}\right)(-2ik)^{2l+1} r^{l+1} \exp(ikr) U\left(l + 1 + \frac{i\gamma}{2k}; 2l + 2; 2ikr\right) \quad (3.10)$$

where ${}_1F_1$(a; b; r)(U(a; b; r)) denote respectively the regular (irregular)confluent hypergeometric functions[18]. The operator $\dot{h}_{l,\gamma}$ has deficiency indices (2,2) and consequently all its self-adjoint extensions may be parameterized by a 4- parameters family of self-adjoint operators[14].

Let us introduce the following two parameters family of s.a extensions of $\dot{h}_{l,\gamma}$ by:

$$h_{l,\gamma,\alpha_l,\beta_l} = -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + \frac{\gamma}{r},$$

$$\mathcal{D}(h_{l,\gamma,\alpha_l,\beta_l}) = \left\{ f \in \mathcal{D}(\dot{h}_{l,\gamma}^*) \left| \begin{array}{l} \left(1 + \frac{\alpha_l}{2}\right) f'(R+) + \left(\frac{\alpha_l}{2} - 1\right) f'(R-) = 0 \\ \left(1 + \frac{\beta_l}{2}\right) f(R+) + \left(\frac{\beta_l}{2} - 1\right) f(R-) = 0. \end{array} \right. \right\},$$
$$\alpha_l, \beta_l \in \mathbb{R}. \quad (3.11)$$

The case $\alpha_l = \beta_l = 0$ in eq. (3.11) yields the free Coulomb hamiltonian $h_{l,\gamma}$ for a fixed angular momentum l.

The two-parameters nonrelativistic δ'-sphere interaction plus Coulomb interaction is defined by

$$H_{\gamma,\alpha,\beta} = \oplus_{l=0}^{\infty} U^{-1} h_{l,\gamma,\alpha_l,\beta_l} U \otimes 1\mathrm{I}, \ \alpha = \{\alpha_l\}_{l=0}^{\infty}, \beta = \{\beta_l\}_{l=0}^{\infty}. \quad (3.12)$$

The particular case $\alpha = \beta = 0$ in eq. (3.12) leads to the Coulomb hamiltonian

$$H_\gamma = \oplus_{l=0}^{\infty} U^{-1} h_{l,\gamma} U \otimes 1\mathrm{I}, \quad (3.13)$$

where

$$h_{l,\gamma} = -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + \frac{\gamma}{r}$$

$$\mathcal{D}(h_{l,\gamma}) = \{f \in L^2((0,\infty))/f, f' \in AC_{loc}((0,\infty));$$
$$-f'' + l(l+1)r^{-2}f + \gamma r^{-1} f \in L^2((0,\infty))\}, \gamma \in \mathbb{R}, l \geq 1. \quad (3.14)$$

The case $\alpha \neq 0, \beta = 0$ in eq. (3.12) yields the nonrelativistic δ'-sphere interaction plus Coulomb interaction of the first type. The case $\beta \neq 0, \alpha = 0$ in eq. (3.12) yields the nonrelativistic δ'-sphere interaction plus Coulomb interaction of the second type [1].

3.2 The resolvent equation

The resolvent of $h_{l,\gamma,\alpha_l,\beta_l}$ and $H_{\gamma,\alpha,\beta}$ are given by the following theorem:

Theorem 3.1 : If $\alpha_l, \beta_l \neq 0$, we have :

(i)The resolvent of $h_{l,\gamma,\alpha_l,\beta_l}$ is given by

$$\left(h_{l,\gamma,\alpha_l\beta_l}-k^2\right)^{-1} = \left(h_{l,\gamma}-k^2\right)^{-1} + \mu_{l,\gamma}(k)\Big[\alpha_l\left(\tilde{\psi}^{(2)}_{l,\gamma}(-\bar{k}),.\right)\psi^{(2)}_{l,\gamma}(k) - \beta_l\left(\psi^{(2)}_{l,\gamma}(-\bar{k}),.\right)\tilde{\psi}^{(2)}_{l,\gamma}(k) - \frac{\alpha_l\beta_l}{2}\left(\psi^{(2)}_{l,\gamma}(-\bar{k}),.\right)\tilde{\psi}^{(1)}_{l,\gamma}(k) + \frac{\alpha_l\beta_l}{2}\left(\tilde{\psi}^{(2)}_{l,\gamma}(-\bar{k}),.\right)\psi^{(1)}_{l,\gamma}(k)\Big],$$
$$k^2 \in \rho\left(h_{l,\gamma,\alpha_l\beta_l}\right);\ \mathrm{Im}\, k>0;\ l \in \mathbb{N}_0 \qquad (3.15)$$

where

$$\mu_{l,\gamma}(k) = \left\{1-\frac{\alpha_l\beta_l}{4} + \frac{1}{2}(\beta_l-\alpha_l)\left[G_{l,\gamma}(k,R)F_{l,\gamma}(k,R)\right]'\right\}^{-1}. \qquad (3.16)$$

$$\psi^{(n)}_{l,\gamma}(k,r) = \begin{cases} G_{l,\gamma}(k,R)F_{l,\gamma}(k,r); & r<R \\ (-1)^n F_{l,\gamma}(k,R)G_{l,\gamma}(k,r), & r>R, n=1,2 \end{cases} \qquad (3.17)$$

$$\tilde{\psi}^{(n)}_{l,\gamma} = \begin{cases} G'_{l,\gamma}(k,R)F_{l,\gamma}(k,r); & r<R \\ (-1)^n F'_{l,\gamma}(k,R)G_{l,\gamma}(k,r), & r>R, n=1,2 \end{cases} \qquad (3.18)$$

The resolvent of $H_{\gamma,\alpha,\beta}$ reads:

(ii) $$\left(H_{\gamma,\alpha,\beta}-k^2\right)^{-1} = \left(H_\gamma-k^2\right)^{-1} + \oplus_{l=0}^{\infty}\oplus_{m=-l}^{l}\mu_{l,\gamma}(k)\Big[\alpha_l\left(|.|\tilde{\psi}^{(2)}_{l,\gamma}(-\bar{k})Y_l^{(m)},.\right)\mathrm{x}\, |.|\psi^{(2)}_{l,\gamma}(k)Y_l^{(m)} - \beta_l\left(|.|\psi^{(2)}_{l,\gamma}(-\bar{k})Y_l^{(m)},.\right)|.|\tilde{\psi}^{(2)}_{l,\gamma}(k)Y_l^{(m)} - \frac{\alpha_l\beta_l}{2}\left(|.|\psi^{(2)}_{l,\gamma}(-\bar{k})Y_l^{(m)},.\right)\mathrm{x}\, |.|\tilde{\psi}^{(1)}_{l,\gamma}(k)Y_l^{(m)} + \frac{\alpha_l\beta_l}{2}\left(|.|\tilde{\psi}^{(2)}_{l,\gamma}(-\bar{k})Y_l^{(m)},.\right)|.|\psi^{(1)}_{l,\gamma}(k)Y_l^{(m)}\Big], k^2 \in \rho\left(H_{\gamma,\alpha,\beta}\right);\ \mathrm{Im}\, k>0,\ l \in \mathbb{N}_0 \qquad (3.19)$$

Proof :
Similar to the proof of theorem 2.1.

Let us now provide the additional information on the domain of $h_{l,\gamma,\alpha_l\beta_l}$ and prove that the two parameters δ' interaction plus Coulomb interaction is local.

Theorem 3.2: The domain $\mathcal{D}\left(h_{l,\gamma,\alpha_l\beta_l}\right), -\infty<\alpha_l,\beta_l\le\infty$ consists of functions of the type:

$$\phi_{l,\gamma,\alpha_l\beta_l}(k,r) = \chi_{l,\gamma}(k,r) + \mu_{l,\gamma}(k)\Big(\alpha_l\chi'_{l,\gamma}(k,R)\psi^{(2)}_{l,\gamma}(k,r) - \beta_l\chi_{l,\gamma}(k,R)\tilde{\psi}^{(2)}_{l,\gamma}(k,r) - \frac{\alpha_l\beta_l}{2}\chi_{l,\gamma}(k,R)\tilde{\psi}^{(1)}_{l,\gamma}(k,r) + \frac{\alpha_l\beta_l}{2}\chi'_{l,\gamma}(k,R)\psi^{(1)}_{l,\gamma}(k,r)\Big) \qquad (3.20)$$

where $\mu_{l,\gamma}(k)$ is defined by eq. (3.16), $\chi_{l,\gamma}\in\mathcal{D}\left(h_{l,\gamma}\right)$ and $k^2\in\rho\left(h_{l,\gamma,\alpha_l\beta_l}\right)$, $\mathrm{Im}k>0$. The decomposition (3.20) is unique and with $\phi_{l,\gamma,\alpha_l\beta_l}$ of this form, we obtain:

$$\left(h_{l,\gamma,\alpha_l\beta_l}-k^2\right)\phi_{l,\gamma,\alpha_l\beta_l} = \left(h_{l,\gamma}-k^2\right)\chi_{l,\gamma}. \qquad (3.21)$$

Consider $\phi_{l,\gamma,\alpha_l\beta_l}\in\mathcal{D}\left(h_{l,\gamma,\alpha_l\beta_l}\right)$ and $\phi_{l,\gamma,\alpha_l\beta_l}=0$ in a open set $\Lambda\subset(0,\infty)$, then $h_{l,\gamma,\alpha_l\beta_l}\phi_{l,\gamma,\alpha_l\beta_l}=0$ in Λ, which means that the interaction described by $h_{l,\gamma,\alpha_l\beta_l}$ is local.

Proof
Similar to the proof of theorem 2.2.

3.3 Spectral properties of $h_{l,\gamma,\alpha_l\beta_l}$

The spectral properties of $h_{l,\gamma,\alpha_l\beta_l}$ are provided by the following theorem where $\sigma_{ess},\sigma_{ac},\sigma_{sc}$ denote the essential spectrum, absolutely continuous spectrum, singularly continuous spectrum respectively.

Theorem 3.3: For all $\alpha_l,\beta_l\in(0,\infty)$, we have: $\sigma_{ess}\left(h_{l,\gamma,\alpha_l\beta_l}\right) = \sigma_{ac}\left(h_{l,\gamma,\alpha_l\beta_l}\right) = [0,\infty)$ (3.22)

$$\sigma_{sc}\left(h_{l,\gamma,\alpha_l\beta_l}\right) = \emptyset. \qquad (3.23)$$

Negative eigenvalues of $h_{l,\gamma,\alpha_l\beta_l}$ are obtained from the equation

$$1-\frac{\alpha_l\beta_l}{4}+\frac{1}{2}(\beta_l-\alpha_l)\psi^{(2)'}_{l,\gamma}\left(\mathrm{i}\sqrt{-E},R\right)=0, E<0. \qquad (3.24)$$

Proof:
Similar to the proof of theorem 2.3.

3.4 Scattering theory for the pair $(h_{l,\gamma,\alpha_l\beta_l}, h_{l,\gamma})$

For $k\ge0$, we define the function

$$\phi_{l,\gamma,\alpha_l\beta_l}(k,r) = F_{l,\gamma}(k,r) + \mu_{l,\gamma}(k)\Big(\alpha_l F'_{l,\gamma}(k,R)\psi^{(2)}_{l,\gamma}(k,r) - \beta_l F_{l,\gamma}(k,R)\tilde{\psi}^{(2)}_{l,\gamma}(k,r) - \frac{\alpha_l\beta_l}{2}F_{l,\gamma}(k,R)\tilde{\psi}^{(1)}_{l,\gamma}(k,r) + \frac{\alpha_l\beta_l}{2}F'_{l,\gamma}(k,R)\psi^{(1)}_{l,\gamma}(k,r)\Big) \qquad (3.25)$$

One can show easily that the function $h_{l,\gamma,\alpha_l\beta_l}$ fulfills the following conditions:

$$\left(1+\frac{\alpha_l}{2}\right)\phi'_{l,\gamma,\alpha_l\beta_l}(R+)+\left(\frac{\alpha_l}{2}-1\right)\phi'_{l,\gamma,\alpha_l\beta_l}(R-)=0$$
$$\left(1+\frac{\beta_l}{2}\right)\phi_{l,\gamma,\alpha_l\beta_l}(R+)+\left(\frac{\beta_l}{2}-1\right)\phi_{l,\gamma,\alpha_l\beta_l}(R-)=0,$$

$$\phi''_{l,\gamma,\alpha_l,\beta_l}(k,r) + l(l+1)r^{-2}\phi_{l,\gamma,\alpha_l,\beta_l}(k,r) + \gamma r^{-1}\phi_{l,\gamma,\alpha_l,\beta_l}(k,r) = k^2\phi_{l,\gamma,\alpha_l,\beta_l}(k,r). \quad (3.26)$$

Consequently, the functions $\phi_{l,\gamma,\alpha_l,\beta_l}$ are the scattering wave functions of $h_{l,\gamma,\alpha_l,\beta_l}$.

For $k > 0$, the study of the asymptotic behavior of $\phi_{l,\gamma,\alpha_l,\beta_l}$ as $r \to \infty$ yields[17]

$$\phi_{l,\gamma,\alpha_l,\beta_l} \underset{r\to\infty}{\longrightarrow} A_{l,\gamma}(k)\sin\left(kr - \frac{\gamma}{2k}\ln(2kr) - \frac{l\pi}{2} + \delta^{(0)}_{l,\gamma}(k)\right) + \mu_{l,\gamma}(k)F'_{l,\gamma}(k,R)F_{l,\gamma}(k,R)B_{l,\gamma}(k)(\alpha_l - \beta_l)\exp\left[-\mathrm{i}\left(kr - \frac{\gamma}{2k}\ln(2kr) - \frac{l\pi}{2} + \delta^{(0)}_{l,\gamma}(k)\right)\right]$$

$$= [A_{l,\gamma}(k) - \mathrm{i}\mu_{l,\gamma}(k)B_{l,\gamma}(k)(\alpha_l - \beta_l)F'_{l,\gamma}(k,R)F_{l,\gamma}(k,R)]\sin x_1 + \mu_{l,\gamma}(k)B_{l,\gamma}(k)(\alpha_l - \beta_l)F'_{l,\gamma}(k,R)F_{l,\gamma}(k,R)\cos x_1 = [L^2_{l,\gamma,1}(k) + L^2_{l,\gamma,2}(k)]^{\frac{1}{2}}\sin\left(x_1 + \delta^{(c)}_{l,\gamma,\alpha_l,\beta_l}(k)\right) + 0(1). \quad (3.27)$$

Where the pure Coulomb phase shift $\delta^{(0)}_{l,\gamma}(k)$ is given by

$$\delta^{(0)}_{l,\gamma}(k) = \arg\left[\Gamma\left(l + 1 + \frac{\mathrm{i}\gamma}{2k}\right)\right] \quad (3.28)$$

and

$$x_1 = kr - \frac{\gamma}{2k}\ln(2kr) - \frac{l\pi}{2} + \delta^{(0)}_{l,\gamma}(k) \quad (3.29)$$

$$L_{l,\gamma,2} = \mu_{l,\gamma}(k)B_{l,\gamma}(k)(\alpha_l - \beta_l)F'_{l,\gamma}(k,R)F_{l,\gamma}(k,R) \quad (3.30)$$

$$L_{l,\gamma,1} = A_{l,\gamma}(k) - \mathrm{i}\mu_{l,\gamma}(k)B_{l,\gamma}(k)(\alpha_l - \beta_l)F'_{l,\gamma}(k,R)F_{l,\gamma}(k,R). \quad (3.31)$$

The Coulomb modified phase shift $\delta^{(c)}_{l,\gamma}(k)$ is given by:

$$\delta^{(c)}_{l,\gamma,\alpha_l,\beta_l}(k) = -\arctan\frac{L_{l,\gamma,2}(k)}{L_{l,\gamma,1}(k)} = -\arctan\frac{\mu_{l,\gamma}(k)B_{l,\gamma}(k)(\alpha_l - \beta_l)F'_{l,\gamma}(k,R)F_{l,\gamma}(k,R)}{A_{l,\gamma}(k) - \mathrm{i}\mu_{l,\gamma}(k)B_{l,\gamma}(k)(\alpha_l - \beta_l)F'_{l,\gamma}(k,R)F_{l,\gamma}(k,R)} \quad (3.32)$$

where [17]

$$A_{l,\gamma}(k) = 2^{-l}k^{-l-1}\exp\left(\frac{\pi\gamma}{4k}\right)\Gamma(2l+2)\left|\Gamma\left(l + 1 + \frac{\mathrm{i}\gamma}{2k}\right)\right|^{-1} \quad (3.33)$$

$$B_{l,\gamma}(k) = \frac{1}{k\,A_{l,\gamma}(k)}. \quad (3.34)$$

The Coulomb modified on-shell scattering matrix is given by:

$$S^{(c)}_{l,\gamma,\alpha_l,\beta_l}(k) = \exp\left(2\mathrm{i}\delta^{(c)}_{l,\gamma,\alpha_l,\beta_l}(k)\right) = 1 - 2\mathrm{i}kB^2_{l,\gamma}(k)\mu_{l,\gamma}(k)(\alpha_l - \beta_l)F'_{l,\gamma}(k,R)F_{l,\gamma}(k,R). \quad (3.35)$$

The corresponding partial wave scattering amplitude reads:

$$f^{(c)}_{l,\gamma,\alpha_l,\beta_l}(k) = \frac{\exp\left(2\mathrm{i}\delta^{(c)}_{l,\gamma,\alpha_l,\beta_l}(k)\right) - 1}{2\mathrm{i}k} = -B^2_{l,\gamma}(k)\mu_{l,\gamma}(k)(\alpha_l - \beta_l)F'_{l,\gamma}(k,R)F_{l,\gamma}(k,R). \quad (3.36)$$

The Coulomb modified effective range expansion corresponding to $h_{l,\gamma,\alpha_l,\beta_l}$ reads [20]

$$(2k)^{2l}\Gamma(2l+2)^{-2}\left|\Gamma\left(l + 1 + \frac{\mathrm{i}\gamma}{2k}\right)\right|^2\exp\left(-\frac{\pi\gamma}{2k}\right)\left(k\cot\delta^{(c)}_{l,\gamma,\alpha_l,\beta_l}(k) - \mathrm{i}k + \exp\left(\frac{\pi\gamma}{2k}\right)h_\gamma(k)\right) = -\frac{1}{a^{(c)}_{l,\gamma,\alpha_l,\beta_l}(k)} + 0(k^2), k > 0, \gamma \in \mathbb{R}. \quad (3.37)$$

where $a^{(c)}_{l,\gamma,\alpha_l,\beta_l}(k)$ is the Coulomb modified scattering length and the function $h_\gamma(k)$ is defined by

$$h_\gamma(k) = \gamma\left|\Gamma\left(1 + \frac{\mathrm{i}\gamma}{2k}\right)\right|^2\left[\frac{\mathrm{i}k}{\gamma} + \ln\left(\frac{2\mathrm{k}}{\mathrm{i}|\gamma|}\right) + \Psi\left(1 + \frac{\mathrm{i}\gamma}{2k}\right)\right]$$

Ψ denotes a digamma function [19].

The Coulomb modified scattering length $a_{l,\gamma,\alpha_l,\beta_l}$ can be calculated from the expansion (3.37).

After a straightforward computation we obtained:

$$-\frac{1}{a^{(c)}_{l,\gamma,\alpha_l,\beta_l}(k)} = \begin{cases} -\dfrac{1 - \frac{\alpha_l\beta_l}{4} + (\beta_l - \alpha_l)\frac{d}{dr}\{rI_\nu(y)k_\nu(y)\}_{r=R}}{\gamma^{-2l-1}\Gamma(2l+2)^2\mathrm{R}^{\frac{1}{2}}\mathrm{I}_\nu(\mathrm{y})(\alpha_l - \beta_l)\frac{d}{dr}\left\{r^{\frac{1}{2}}I_\nu(y)\right\}_{r=R}}, & \gamma \geq 0 \\ -\dfrac{1 - \frac{\alpha_l\beta_l}{4} - \mathrm{i}\frac{\pi}{2}(\beta_l - \alpha_l)\frac{d}{dr}\{rk_\nu(z)H_\nu(z)\}_{r=R}}{\gamma^{-2l-1}\Gamma(2l+2)^2\mathrm{R}^{\frac{1}{2}}\mathrm{J}_\nu(\mathrm{z})(\alpha_l - \beta_l)\frac{d}{dr}\left\{r^{\frac{1}{2}}J_\nu(z)\right\}_{r=R}}, & \gamma \leq 0 \end{cases} \quad (3.38)$$

where we have used the notations $\nu = 2l + 1, y = (4\gamma r)^{\frac{1}{2}}$ and, $z = (4|\gamma|r)^{\frac{1}{2}}$.

The corresponding differential scattering cross section is

$$\sigma = 4\sum_{l=0}^{\infty}\left(l + \frac{1}{2}\right)^2\left|f^{(c)}_{l,\gamma,\alpha_l,\beta_l}(k)\right|^2 P_l^2\cos\theta \quad (3.39)$$

A straightforward computation shows that the total scattering cross section for the pair $(h_{l,\gamma,\alpha_l,\beta_l}, h_{l,\gamma})$ reads:

$$\sigma_{total} = 8\pi \sum_{l=0}^{\infty} \left(l+\frac{1}{2}\right) \left|f_{l,\gamma,\alpha_l,\beta_l}^{(c)}(k)\right|^2$$

$$= 8\pi \sum_{l=0}^{\infty} \left(l + \frac{1}{2}\right) \left| \frac{B_{l,\gamma}^2(k)(\alpha_l-\beta_l)F_{l,\gamma}'(k,R)F_{l,\gamma}(k,R)}{1-\frac{\alpha_l\beta_l}{4}+\frac{1}{2}(\beta_l-\alpha_l)\left[G_{l,\gamma}(k,R)F_{l,\gamma}(k,R)\right]'} \right|^2 \quad (3.41)$$

Or

$$\sigma_{total} = \sum_{l=0}^{\infty} \sigma_l \quad (3.42)$$

where the partial cross sections σ_l is given by :

$$\sigma_l = 4\pi(2l+1)k^{-2}\sin^2\delta_{l,\gamma,\alpha_l,\beta_l}^{(c)}(k). \quad (3.43)$$

Conclusion

In this paper, using the self-adjoint theory of symmetric operator in Hilbert space, we studied the basic properties of two parameters nonrelativistic δ'-sphere and δ'-sphere plus Coulomb (where a charged particle is perturbed by a δ'-sphere interaction). For the both interactions we obtain interesting results on:

- resolvent equations
- spectral properties
- Scattering data (the phase shift, scattering matrix, scattering amplitude, scattering cross section).

As perspective, one can study the case where δ'-sphere interaction is centered on finitely many concentric spheres.

References

[1] Hounkonnou M.N., Hounkpe M., Shabani J., 1999. Exactly solvable models of δ'-sphere interactions in nonrelativistic quantum mechanics, J. Math. Phys.**40**, 4254-4273.

[2] Antoine J.P., Gesztesy F. and Shabani J., 1988. Exactly solvable models of sphere interactions in quantum mechanics, J. Phys. **A20**, 3687-3712.

[3] Shabani J., 1988. Finitely many δ-interactions with supports on concentric sphere, J. Math. Phys. **29**, 660-664.

[4] Dabrowski L. and Shabani J., 1988. Finitely many sphere interactions in quantum mechanics nonseparated boundary conditions, J. Math. Phys. **29**, 2241-2244.

[5] Antoine J.P., Exner P., Seba P. and Shabani J., 1994. A mathematical model of heavy quarkonia mesonic decays, Ann. Phys. (N.Y) **233**,1-16.

[6] Shabani J., 1988. Some properties of the hamiltonian describing a finite number of δ'-interactions with supports on centric spheres, Nuovo Cimento **B101**, 429-439.

[7] Hounkonnou M.N., Hounkpe and Shabani J., 1997. Scattering theory for finitely many sphere interactions supported by concentric spheres, J. Math. Phys. **38**, 2832-2850.

[8] Dittrich J., Exener P. and Sěba P., 1989. Dirac operators with a spherically symmetric δ −shell interaction, J. Math. Phys.**30**, 2875-2882.

[9] Dittrich J., Exner P. and Sěba P., 1992. Dirac Hamiltonian with Coulomb potential and spherically symmetric shell contact interaction, J. Math. Phys. **33**, 2207-2214.

[10] Shabani J. and Vyabandi A., 2002. Exactly solvable models of relativistic δ -sphere interactions, J. Math. Phys. **43**, 6064-6084.

[11] Green I.M. and Moszkowski S.A., 1965. Nuclear coupling schemes with a surface delta interaction, Phys. Rev **139 B**, 790.

[12] Blinder S.M., 1978. Modified delta-function potential for hyperfine interactions, Phys. Rev. **A18**, 853.

[13] Rubio J. and Garcia- Moliner F., 1967. Formal theory of equivalent potentials in solids II. Scattering theory approach for muffin-tin potentials, Proc. Phys. Soc **92**, 206.

[14] Akhiezer W.I. and Glazman I.M., 1981. Theory of linear operators in Hilbert space. Vol.2, Pitman publishing Inc. Boston, London, Melbourne.

[15] Dittrich J., Exner P. and Sěba P., 1989. Dirac operators with a spherically symmetric δ-shell interaction, J. Math. Phys. **30**, 2875-2882.

[16] Reed M. and Simon B., 1978. Methods of modern mathematical physics, Vol.4. Analysis of operators (Academic, New York, 1978).

[17] Gesztesy F. and Thaller B., 1981. Born expressions for Coulomb-type interactions, J. Phys. **A14**, 639-657.

[18] Newton R.G., 1966. Scattering theory of waves and particles, Mcgraw-Hill, Inc. New York San Francisco

[19] Abramowitz M. and Stegurn I.A., 1972. Handbook of Mathematical functions, Dover publication Inc, New-York N.Y.

[20] Bolle D. and Gesztesy F., 1984. Scattering observables in arbitrary dimension n≥2, Phys. Rev. **A30**, 1279-1293.